9. THE DT AND THE CT:

The Dissection Theorem and the Chain Theorem

How to find the gain of a multistage amplifier as the product of separately calculated low entropy factors

Null Double Injection (ndi)

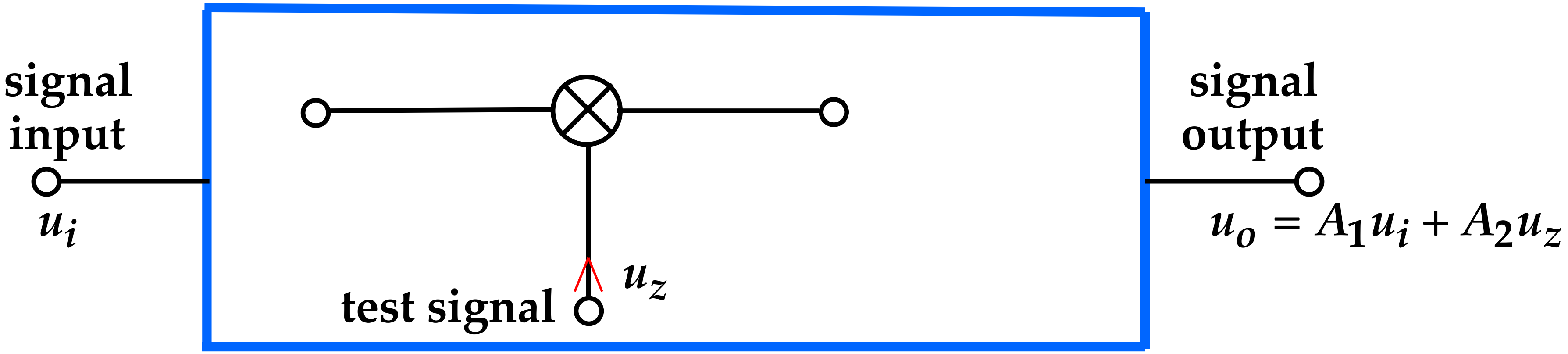
Usually, a transfer function (TF) is calculated as a response to a single independent excitation.

However, large analysis benefits accrue when certain constraints are imposed on several excitations present simultaneously.

For any linear system model:

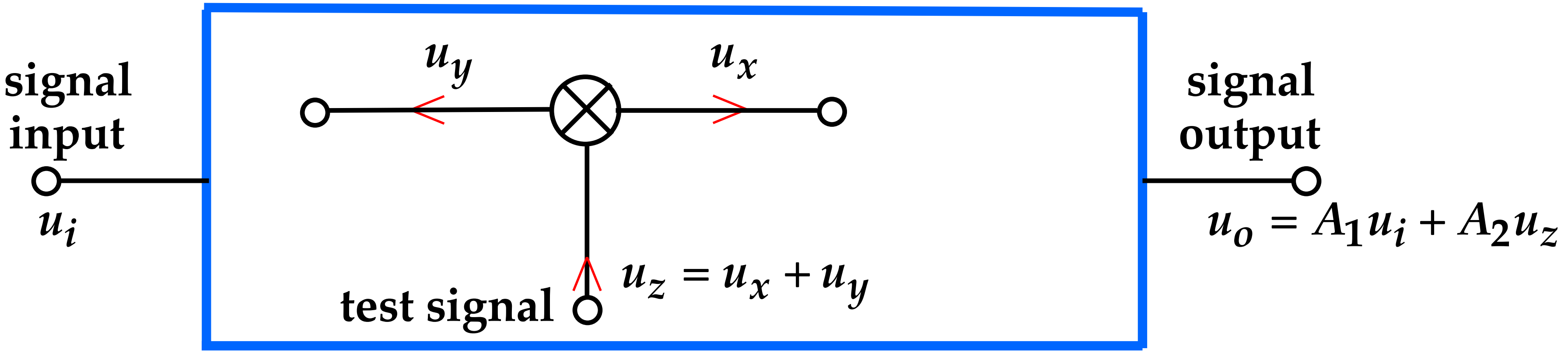
The input is an independent signal, the output is a proportional dependent signal.

Consider a second input, an injected "test signal" u_z :



Since the model is linear, the output is now a linear sum of the values it would have with each input alone.

There are now two more dependent signals, u_x and u_y, where $u_x + u_y = u_z$:



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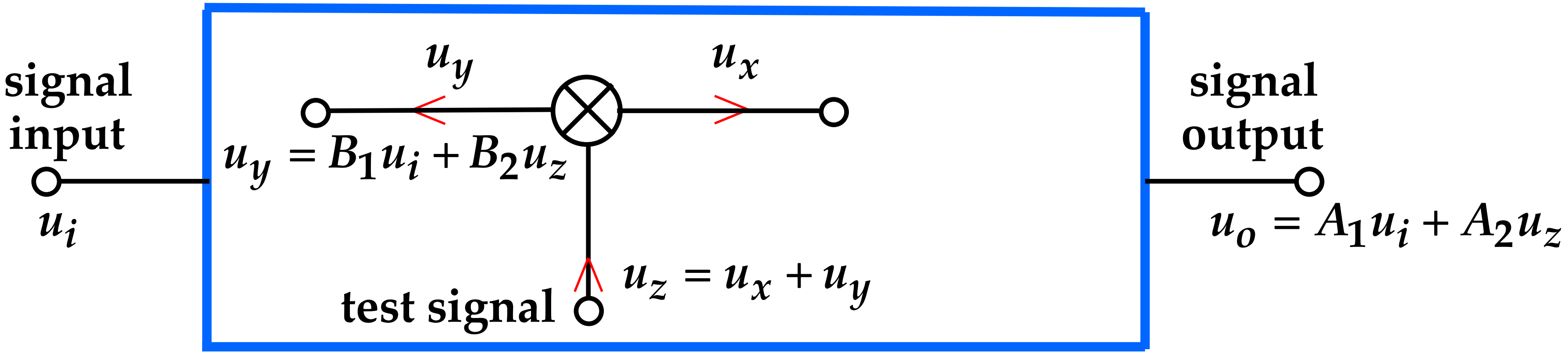

The dependent signal u_y is also a linear sum of the values it would have with each input alone.

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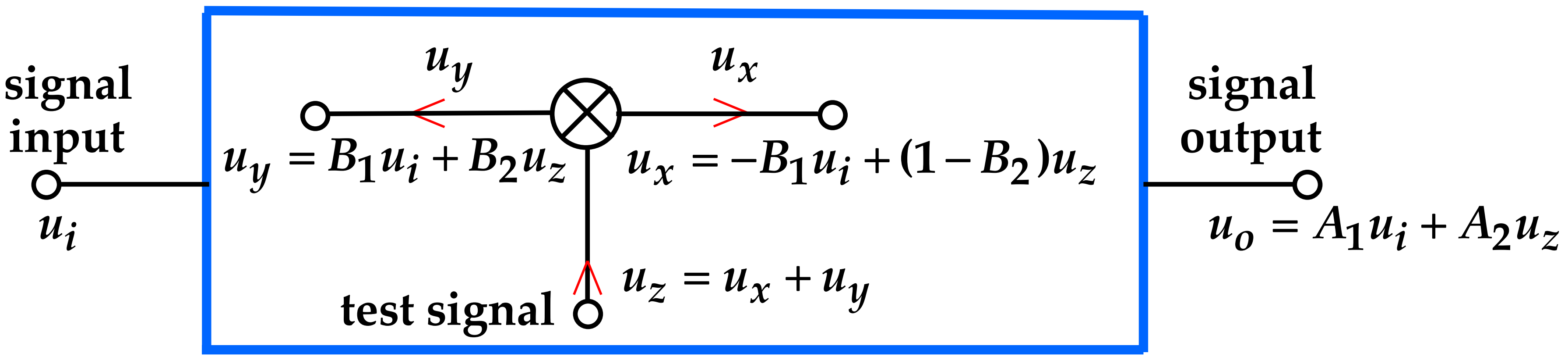


signal input

u_i

u_y

u_x

$u_y = B_1 u_i + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_z = u_x + u_y$

test signal

signal output

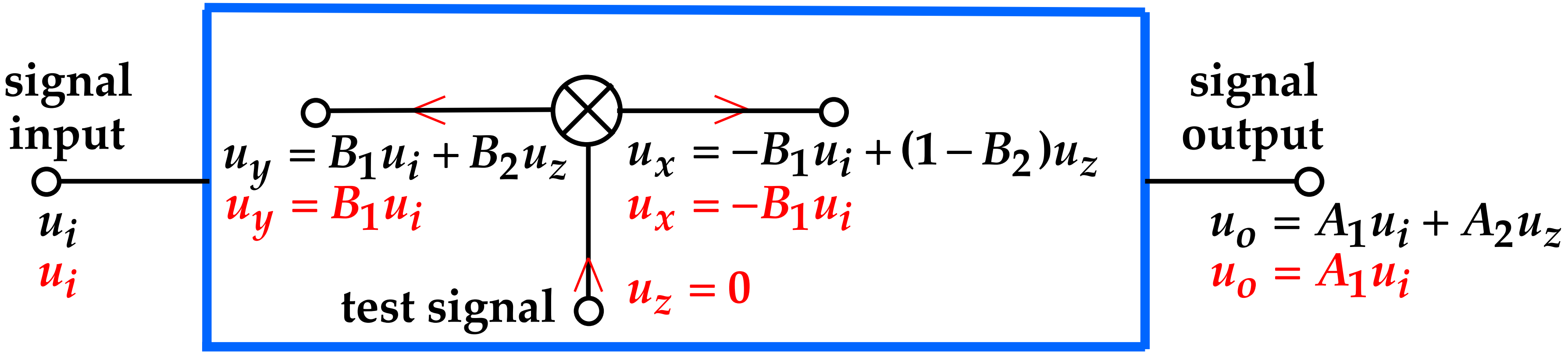
$u_o = A_1 u_i + A_2 u_z$

The dependent signal u_y is also a linear sum of the values it would have with each input alone.

By virtue of $u_z = u_x + u_y$, the independent signal u_x can also be expressed in terms of B_1 and B_2.

Several transfer functions (TFs) can be defined:

Special case 1: $u_z = 0$



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_y = B_1 u_i$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_x = -B_1 u_i$

test signal

$u_z = 0$

signal output

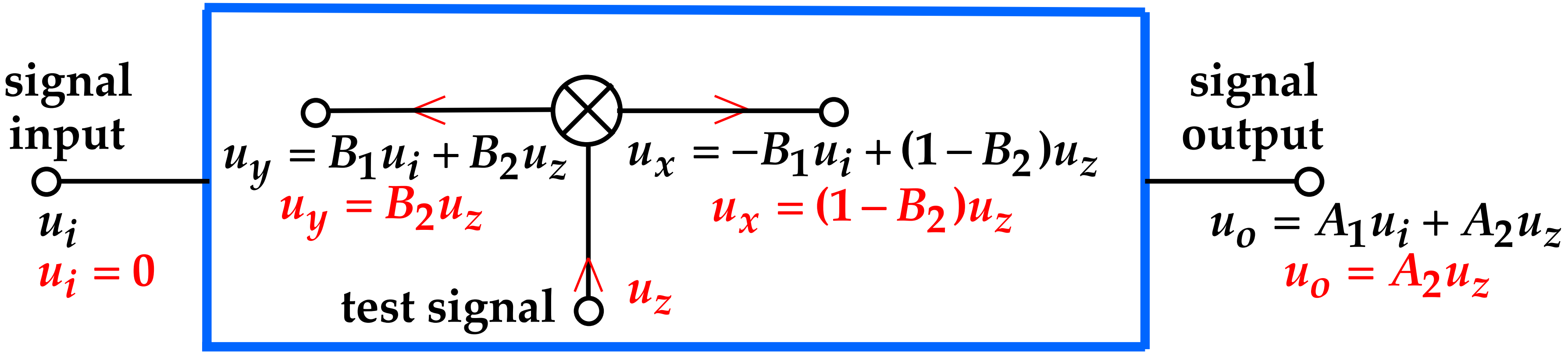
$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i$

$$H \equiv \left. \frac{u_o}{u_i} \right|_{u_z = 0} = A_1$$

Several transfer functions (TFs) can be defined:

Special case 2: $u_i = 0$



u_i

$u_i = 0$

$$u_y = B_1 u_i + B_2 u_z$$

$$u_y = B_2 u_z$$

$$u_x = -B_1 u_i + (1 - B_2) u_z$$

$$u_x = (1 - B_2) u_z$$

$$u_o = A_1 u_i + A_2 u_z$$

$$u_o = A_2 u_z$$

$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0} = A_1$$

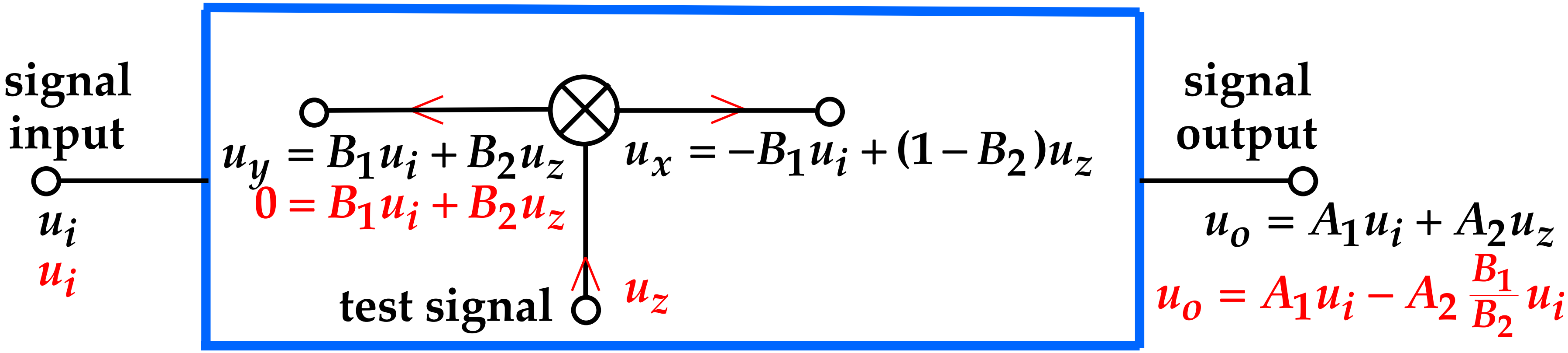
$$T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0} = \frac{B_2}{1 - B_2}$$

These are *single injection* (si) TFs

Several transfer functions (TFs) can be defined:

Special case 3: $u_y = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_y



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$0 = B_1 u_i + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

test signal

u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i - A_2 \frac{B_1}{B_2} u_i$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = \underset{\uparrow}{A_1} - \underset{\uparrow}{A_2 \frac{B_1}{B_2}}$$

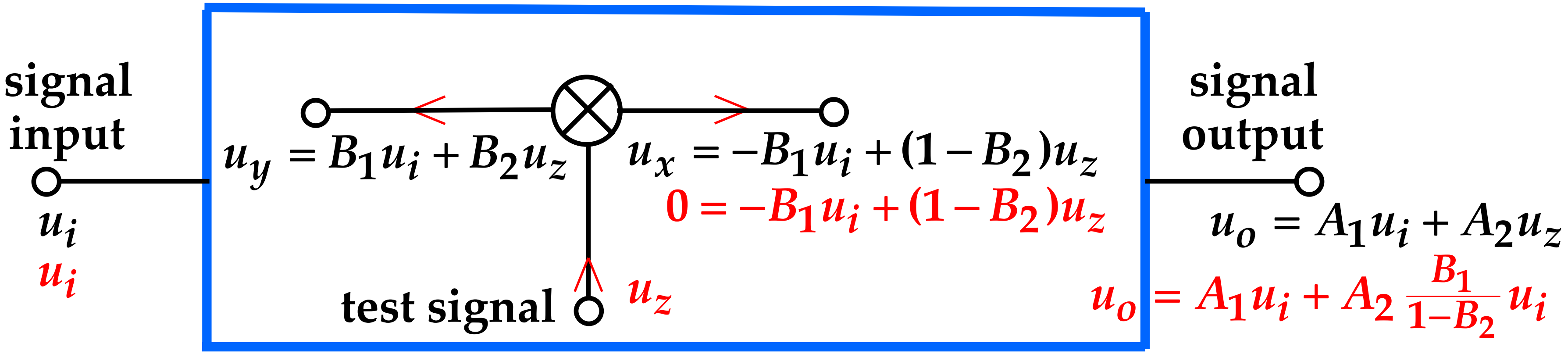
component of u_o from u_i

component of u_o from u_z
adjusted to null u_y

Several transfer functions (TFs) can be defined:

Special case 4: $u_x = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_x



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$0 = -B_1 u_i + (1 - B_2) u_z$

test signal u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i + A_2 \frac{B_1}{1-B_2} u_i$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = \underset{\uparrow}{A_1} + A_2 \underset{\uparrow}{\frac{B_1}{1-B_2}}$$

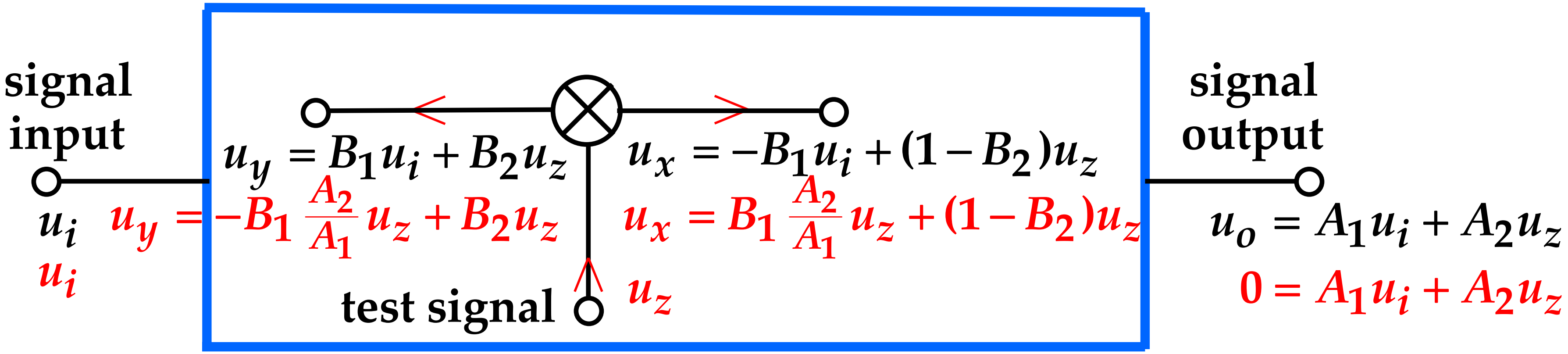
component of u_o from u_i

component of u_o from u_z
adjusted to null u_x

Several transfer functions (TFs) can be defined:

Special case 5: $u_o = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_o



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_y = -B_1 \frac{A_2}{A_1} u_z + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_x = B_1 \frac{A_2}{A_1} u_z + (1 - B_2) u_z$

test signal

u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$0 = A_1 u_i + A_2 u_z$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

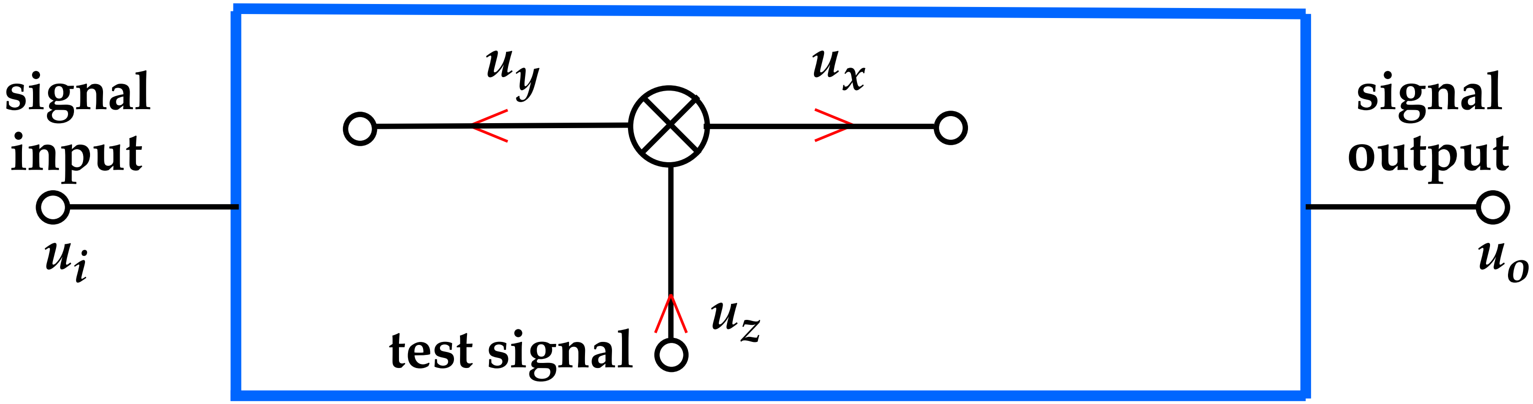
$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2}$$

$$T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0} = \frac{A_1 B_2 - A_2 B_1}{A_1 - (A_1 B_2 - A_2 B_1)}$$

These are *null double injection* (ndi) TFs

Assembled results, so far:

First level TF: $H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0} = A_1$ **(si)**



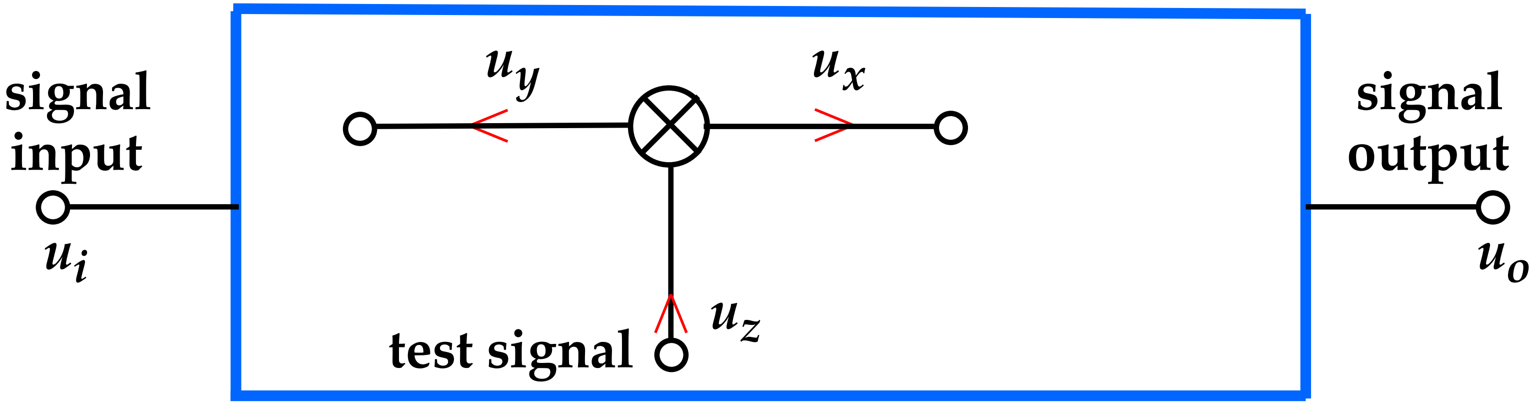
Second level TFs:

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2\frac{B_1}{B_2} \quad \textbf{(ndi)} \qquad T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0} = \frac{A_1B_2 - A_2B_1}{A_1 - (A_1B_2 - A_2B_1)} \quad \textbf{(ndi)}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = A_1 + A_2\frac{B_1}{1-B_2} \quad \textbf{(ndi)} \qquad T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0} = \frac{B_2}{1-B_2} \quad \textbf{(si)}$$

Note that A_2 and B_1 occur only as a product A_2B_1.

The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the A's and B's.



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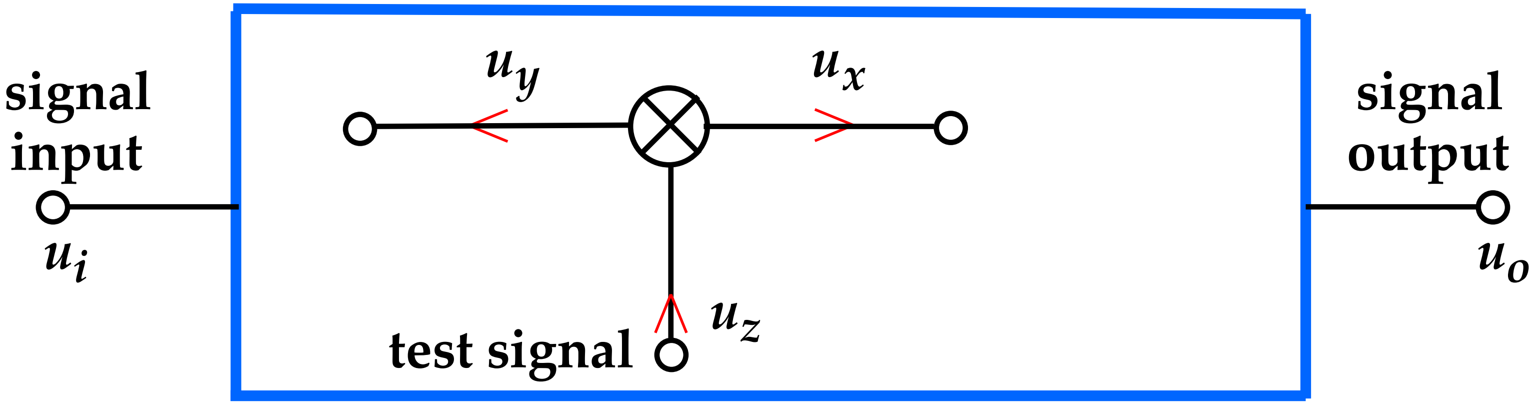

The 4 second level TFs are defined in terms of the 4 original parameters A_1, A_2, B_1, B_2. Since A_2 and B_1 occur only as a product $A_2 B_1$, there are actually only 3 parameters and there must be a relation between the 4 second level TFs, which is

Redundancy Relation:
$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

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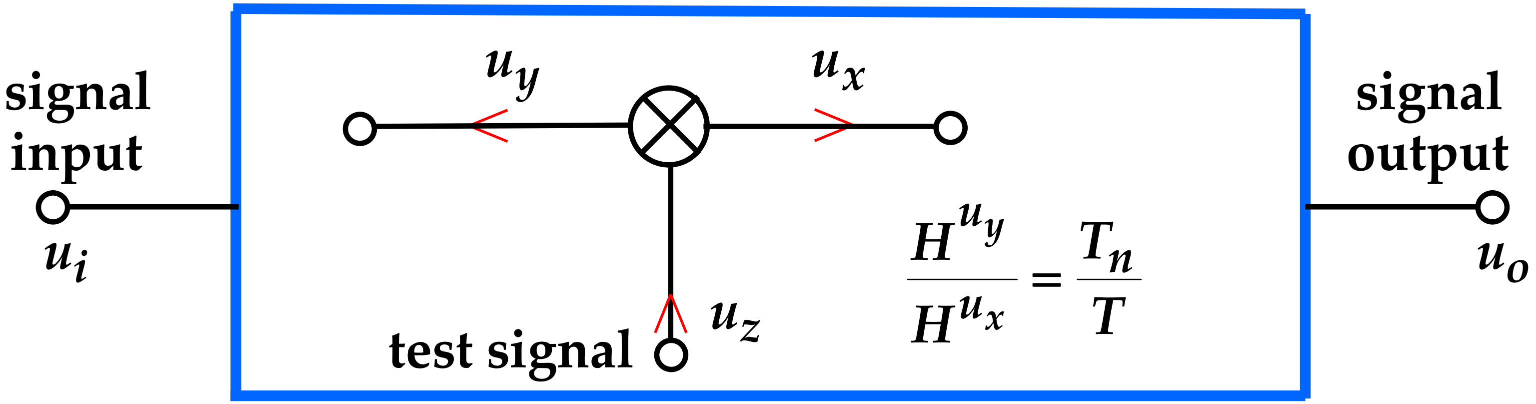


$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

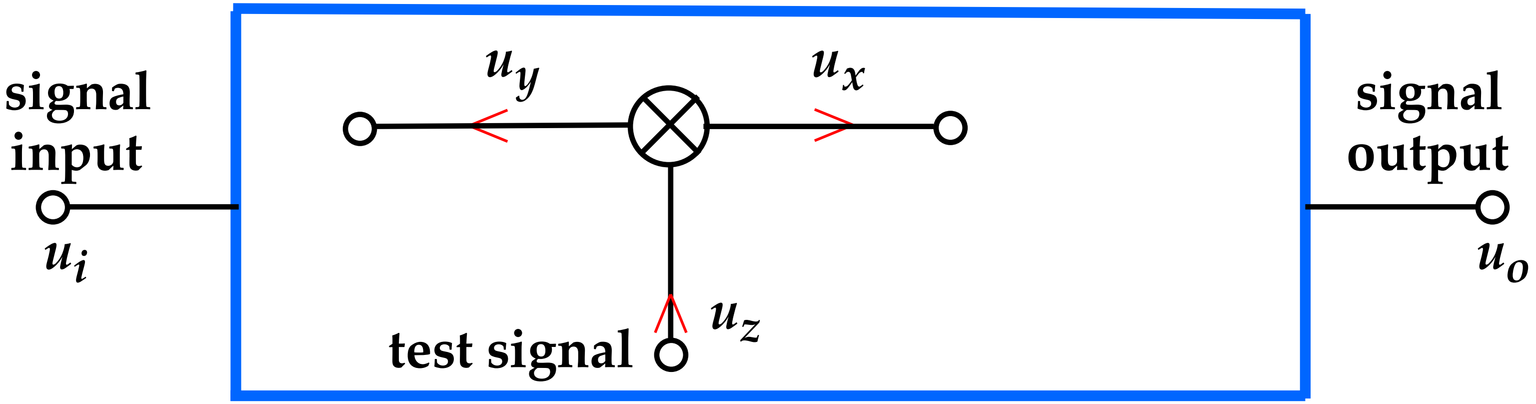
A consequence of the Redundancy Relation is that the first level TF H can be expressed in terms of any three of the four second level TFs H^{u_y}, H^{u_x}, T_n, T.

Two useful versions are:

$$H = H^{u_y}\frac{1+\frac{1}{T_n}}{1+\frac{1}{T}} \qquad H = H^{u_y}\frac{T}{1+T} + H^{u_x}\frac{1}{1+T}$$

These two versions, and the redundancy relation, can easily be verified by substitution of the definitions. After this, the *A*'s and *B*'s are no longer required, and will not appear again.

Dissection Theorem (DT)



ndi ndi ndi

$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0} \qquad H = H^{u_y}\frac{1+\frac{1}{T_n}}{1+\frac{1}{T}} = \underbrace{H^{u_y}\frac{T}{1+T} + H^{u_x}\frac{1}{1+T}}_{\text{second level TFs}}$$

si

first level TF

Notation:
Superscript signal is signal being nulled

$$\overbrace{H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} \qquad T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0}}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} \qquad T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0}$$

Redundancy Relation:

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

These results constitute the Dissection Theorem (DT), so named because it shows that a first level TF can be "dissected" into three second level TFs established in terms of an injected test signal.

The DT is completely general, and applies to any TF
of a linear system model.

For example, H could be a voltage gain, current gain, or an input or output impedance.

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

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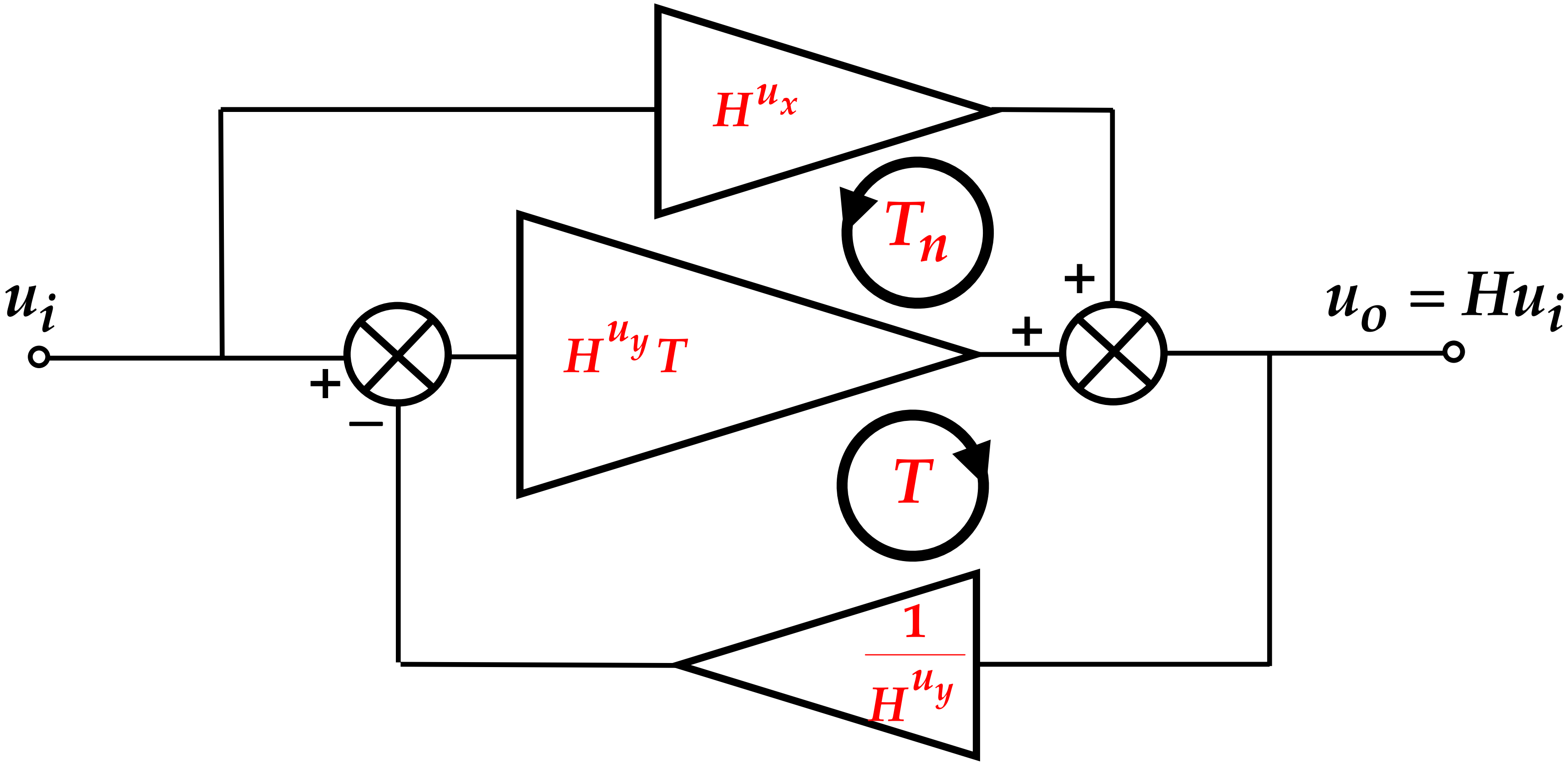
Why are ndi calculations *always* simpler and easier than si calculations?

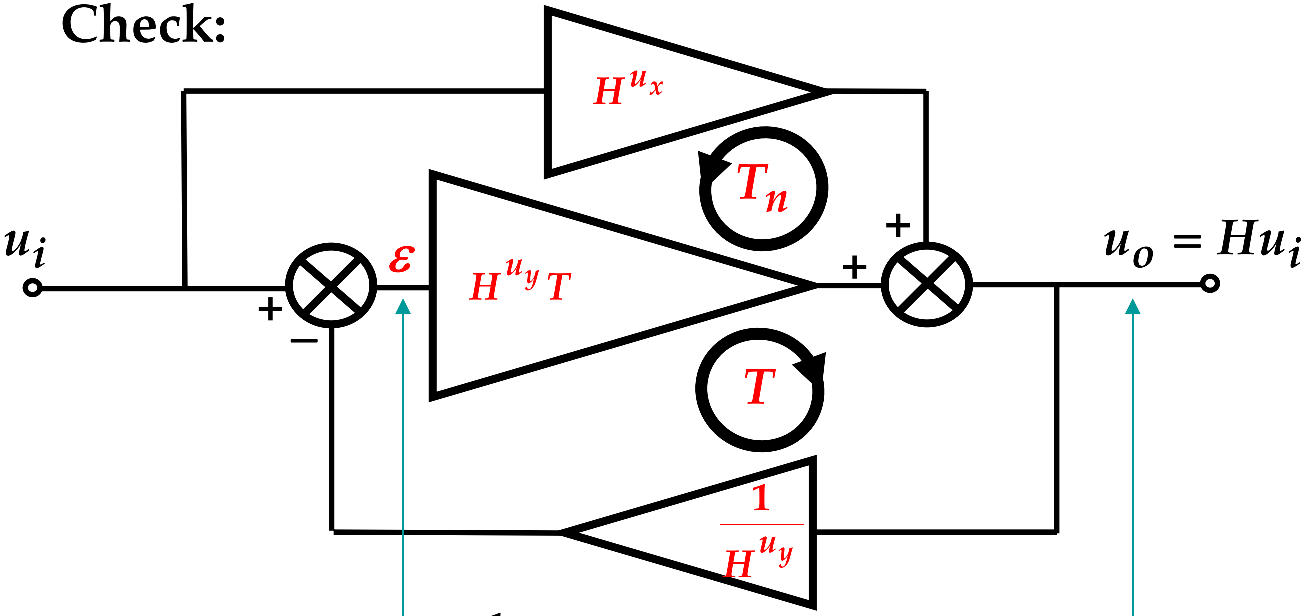
Because any element that supports a null signal does not contribute to the result, and because if one signal is nulled, often other signals are automatically nulled as well, and therefore several elements may be absent from the result.

The Dissection Theorem can be represented by the block diagram



Important: **The individual blocks do not necessarily represent identifiable parts of the actual circuit!**

Check:



$$T_n = H^{u_y}T\frac{1}{H^{u_x}}$$

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

$$\varepsilon = u_i - \frac{1}{H^{u_y}}u_o$$

$$u_o = H^{u_y}T\varepsilon + H^{u_x}u_i$$

$$u_o = H^{u_y}T\left(u_i - \frac{1}{H^{u_y}}u_o\right) + H^{u_x}u_i$$

$$(1+T)u_o = \left(H^{u_y}T + H^{u_x}\right)u_i$$

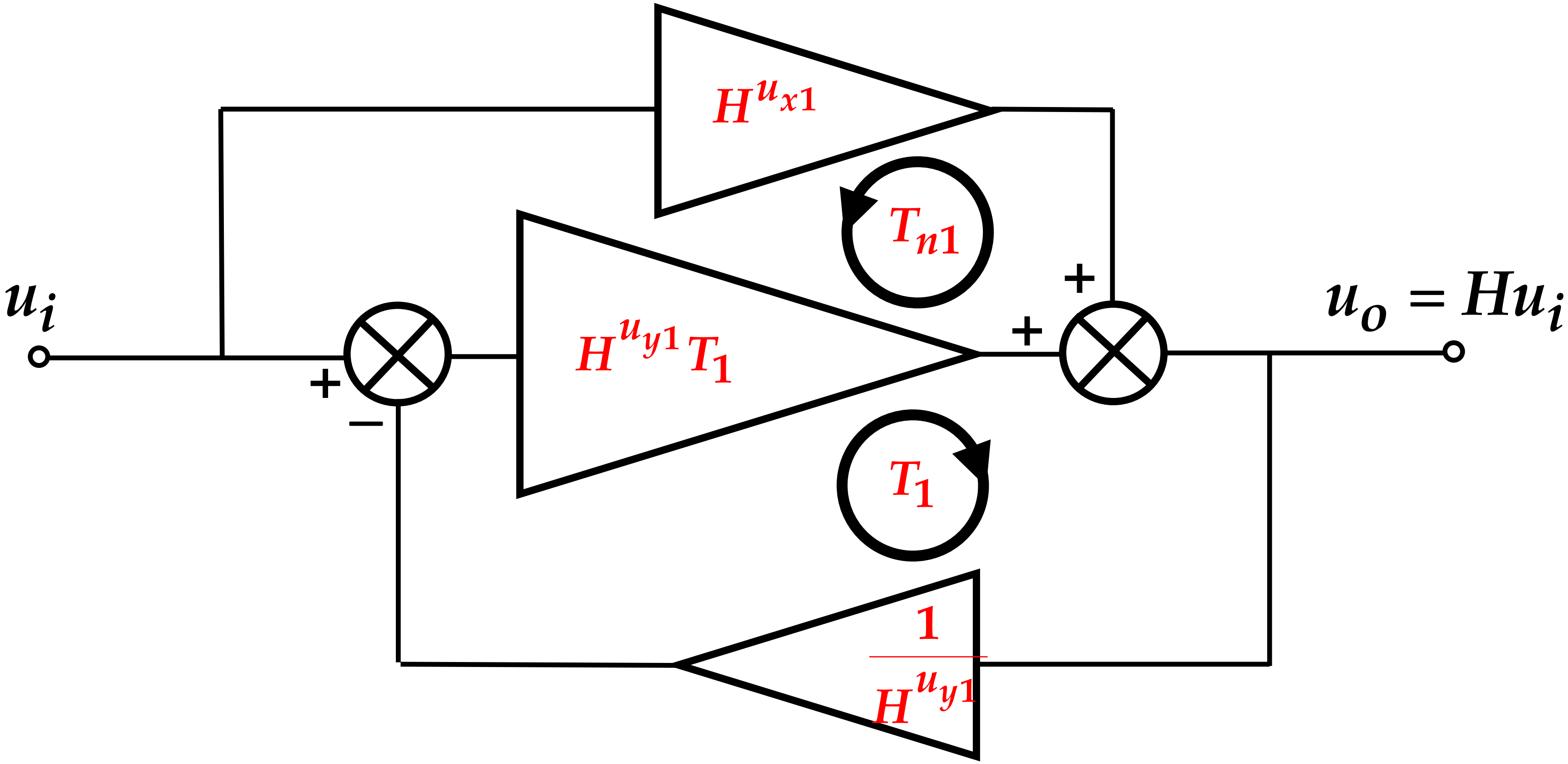
$$H = H^{u_y}\frac{T}{1+T} + H^{u_x}\frac{1}{1+T}$$

So far, nothing has been said about *where* in the system model the test signal is injected.

Different test signal injection points define different sets of second level TFs. Nevertheless, when a mutually consistent set is substituted into the DT, *the same H* results:

$$H = H^{u_{y1}} \frac{1+\frac{1}{T_{n1}}}{1+\frac{1}{T_1}} = H^{u_{y2}} \frac{1+\frac{1}{T_{n2}}}{1+\frac{1}{T_2}} = \ldots$$

This means that the blocks in the block diagram have different values for different test signal injection points:



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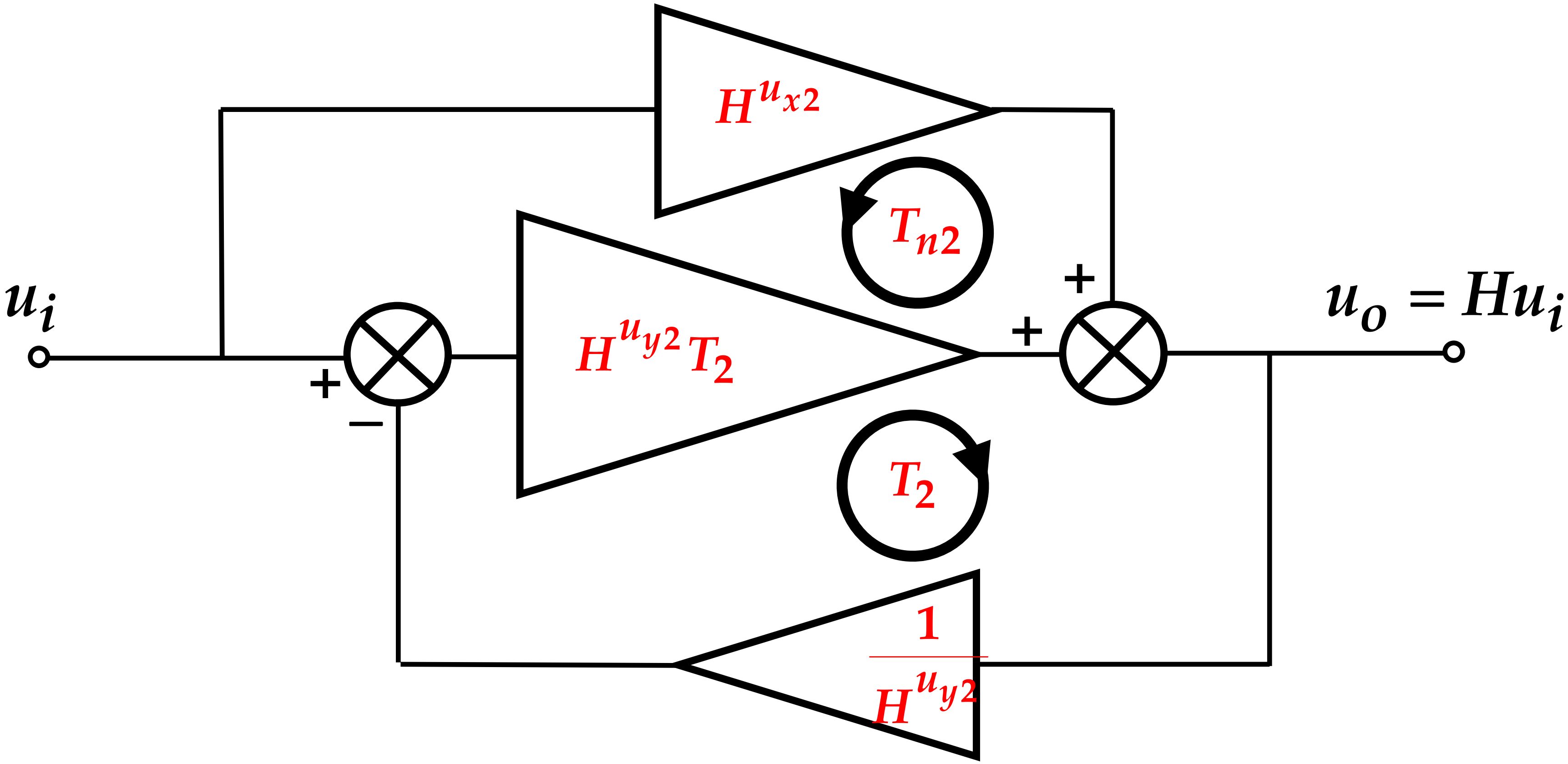


Important: **The individual blocks do not necessarily represent identifiable parts of the actual circuit!**

Not only does the DT implement the Design & Conquer objective, but the DT is itself a Low Entropy Expression, and *much greater* benefits accrue if the second level TFs have useful physical interpretations.

Thus, the second level TFs themselves contain the useful design-oriented information and you may never need to actually substitute them into the theorem.

For example, if $T, T_n >> 1$, $H \approx H^{u_y}$

How to determine the physical interpretations of the second level TFs?

What kind of signal (voltage or current) is injected, and where it is injected, defines an "injection configuration."

Therefore, ***the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.***

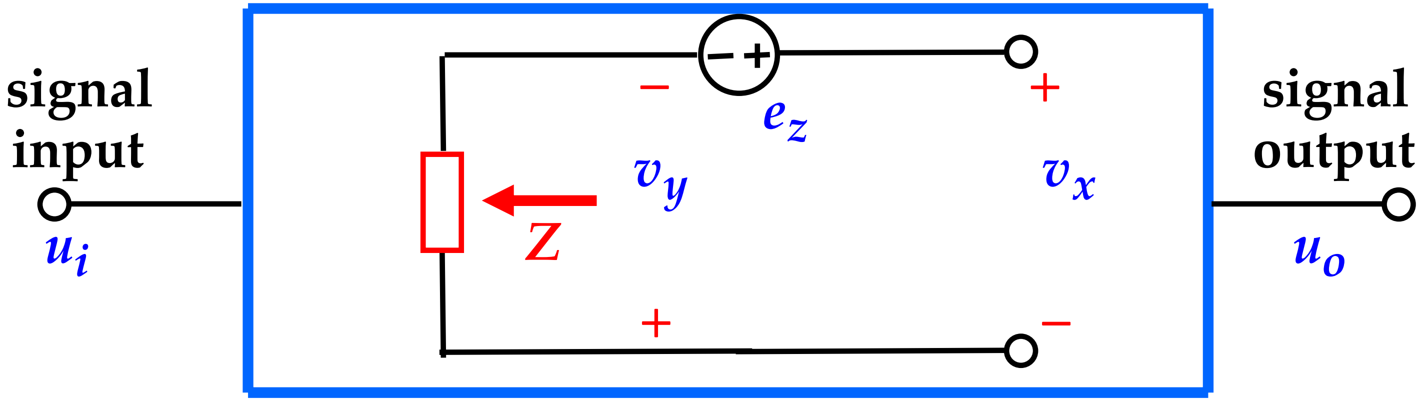
Specific injection configurations for the DT lead to the:

- **Extra Element Theorem (EET)**
- **Chain Theorem (CT)**
- **General Feedback Theorem (GFT)**

As usual, dual forms of the theorem emerge depending upon whether the injected signal u_z is a voltage or a current.

The Extra Element Theorem

Inject a test voltage e_z in series with an element Z such that v_y appears across Z:



The DT is:

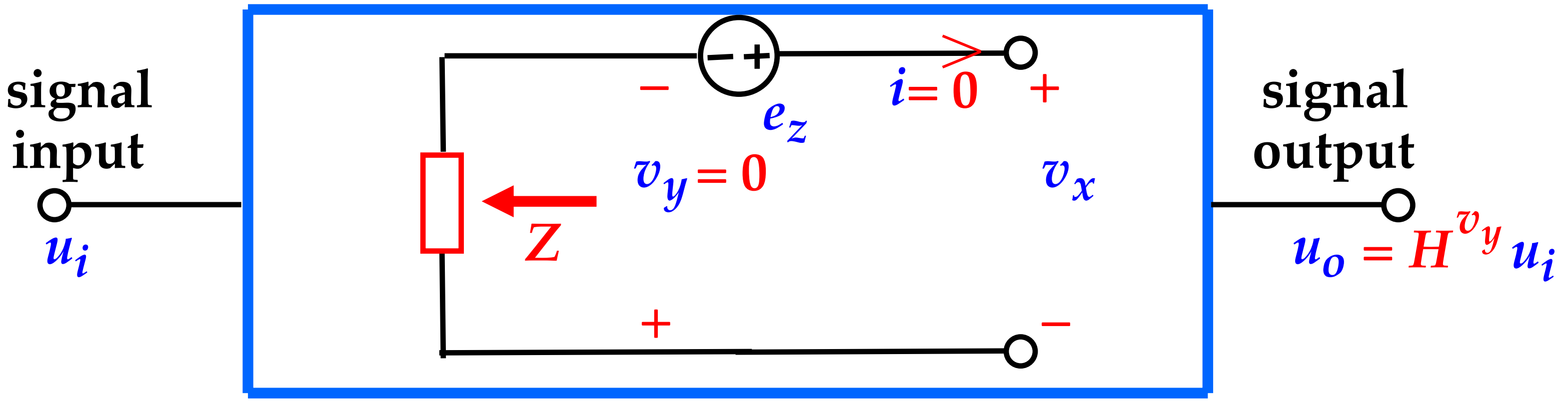
$$H = H^{v_y} \frac{1 + \frac{1}{T_{nv}}}{1 + \frac{1}{T_v}}$$

where:

$$H^{v_y} \equiv \left. \frac{u_o}{u_i} \right|_{v_y=0} \qquad T_v \equiv \left. \frac{v_y}{v_x} \right|_{u_i=0} \qquad T_{nv} \equiv \left. \frac{v_y}{v_x} \right|_{u_o=0}$$

The Extra Element Theorem

To find H^{v_y}, assume that e_z and u_i have been mutually adjusted to null v_y :



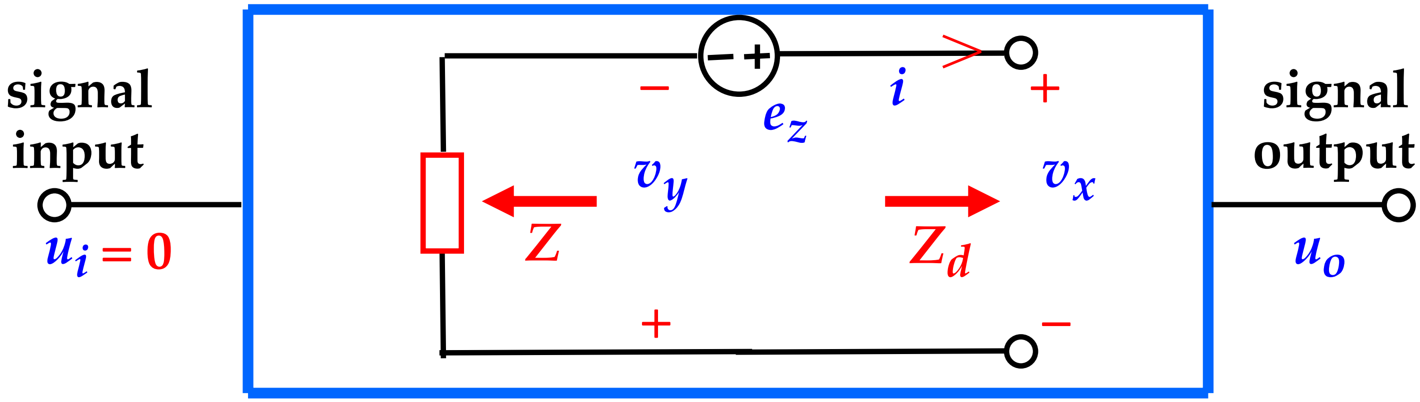
If $v_y = 0$, there is no current through Z, and so the current i into the test port is also zero, which is the condition that would exist if there were no injected test signal and Z were open. Therefore,

$$H\big|_{Z=\infty} = H^{v_y} \equiv \left.\frac{u_o}{u_i}\right|_{v_y=0}$$

where $H\big|_{Z=\infty}$ is the first level TF H when $Z = \infty$.

The Extra Element Theorem

To find T_v, set $u_i = 0$:



The si driving point impedance Z_d looking into the test port is

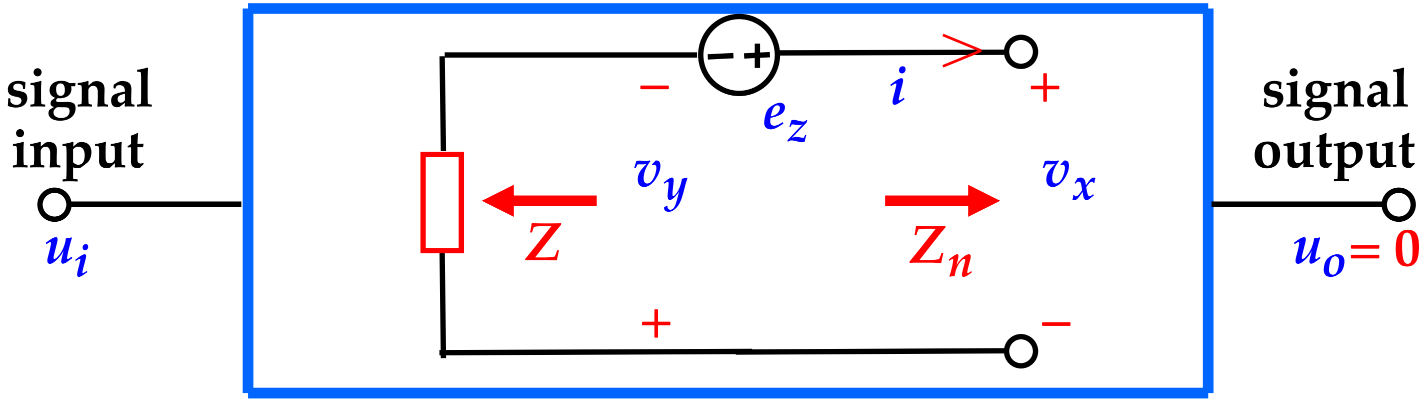
$$Z_d \equiv \left.\frac{v_x}{i}\right|_{u_i=0}$$

Since Z and Z_d are in series with the same current i,

$$T_v = \left.\frac{v_y}{v_x}\right|_{u_i=0} = \frac{Z}{Z_d}$$

The Extra Element Theorem

To find T_{nv}, assume that e_z and u_i have been mutually adjusted to null u_o :



The ndi driving point impedance Z_n looking into the test port is

$$Z_n \equiv \left.\frac{v_x}{i}\right|_{u_o=0}$$

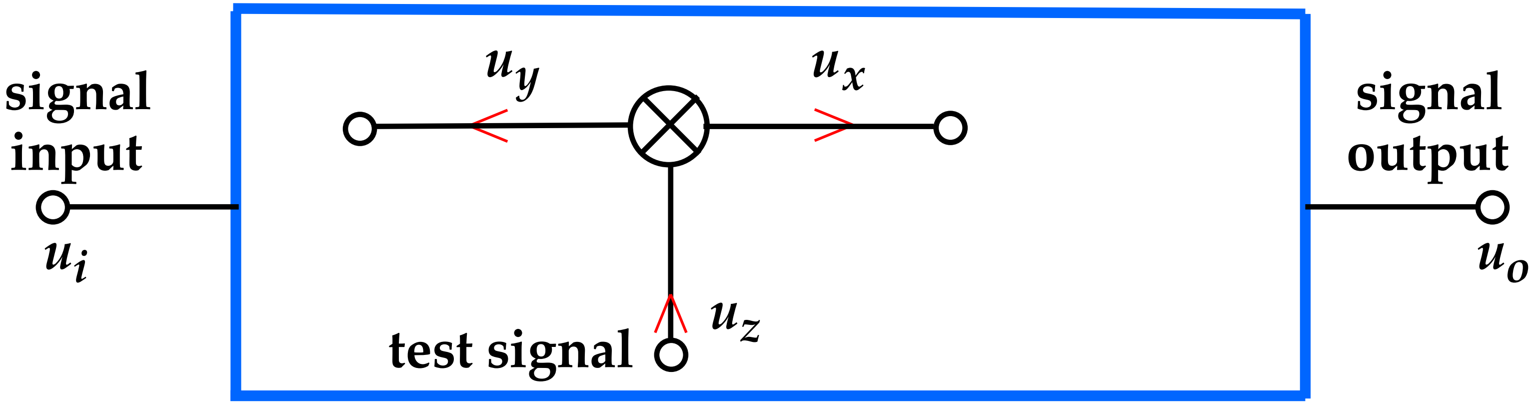
Since Z and Z_n are in series with the same current i,

$$T_{nv} = \left.\frac{v_y}{v_x}\right|_{u_o=0} = \frac{Z}{Z_n}$$

With the second level TFs replaced by the new definitions, the DT morphs into the Extra Element Theorem (EET):

$$H = H\big|_{Z=\infty}\frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}}$$

Dissection Theorem (DT)



$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0}$$

$$H = \overset{\text{ndi}}{H^{u_y}}\,\frac{1+\overset{\text{ndi}}{\frac{1}{T_n}}}{1+\underset{\text{si}}{\frac{1}{T}}} = \underbrace{H^{u_y}\frac{T}{1+T} + \overset{\text{ndi}}{H^{u_x}}\frac{1}{1+T}}$$

first level TF

second level TFs

$$\overbrace{\qquad\qquad\qquad\qquad}$$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} \qquad T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} \qquad T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0}$$

Notation:

Superscript signal is signal being nulled

Redundancy Relation:

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

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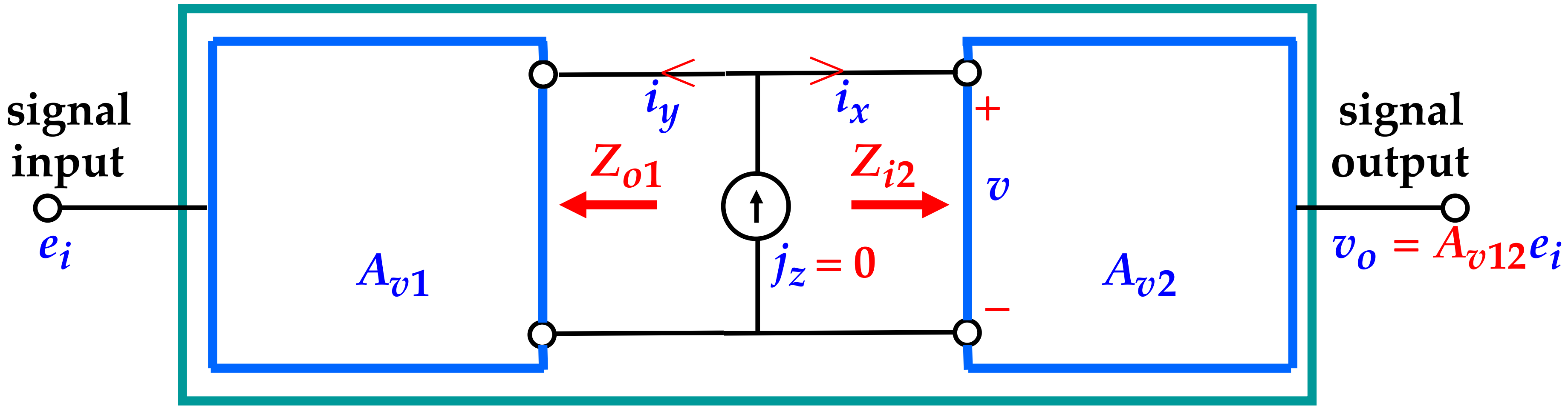
Therefore, ***the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.***

Another special case of the DT leads to the Chain Theorem (CT).

The test signal injection configuration is such that the entire signal from the input flows to the output (no bypass paths).

The Chain Theorem (CT)



The gain $A_{v12} \equiv \frac{v_o}{e_i}$ is given by the DT:

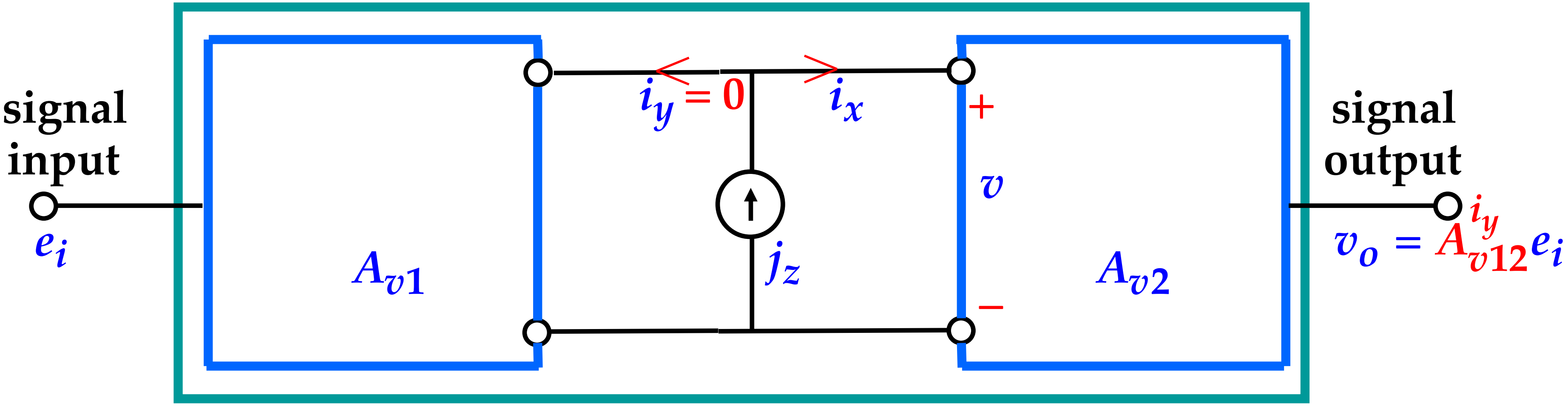
$$A_{v12} = A_{v12}^{i_y} \frac{1+\frac{1}{T_{ni}}}{1+\frac{1}{T_i}}$$

The TF $T_{ni} \equiv i_y/i_x \big|_{v_o=0}$ is an ndi calculation with the output v_o nulled.

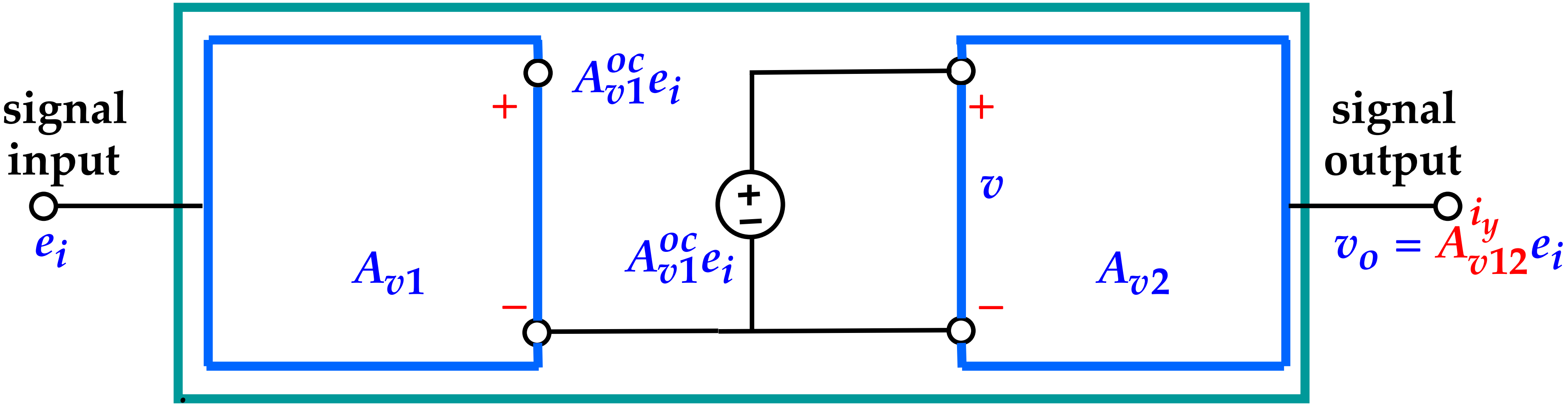
If v_o is nulled, so is i_x, so $T_{ni} = \infty$.

This implies that T_{ni} is infinite unless the signal can bypass the injection point.

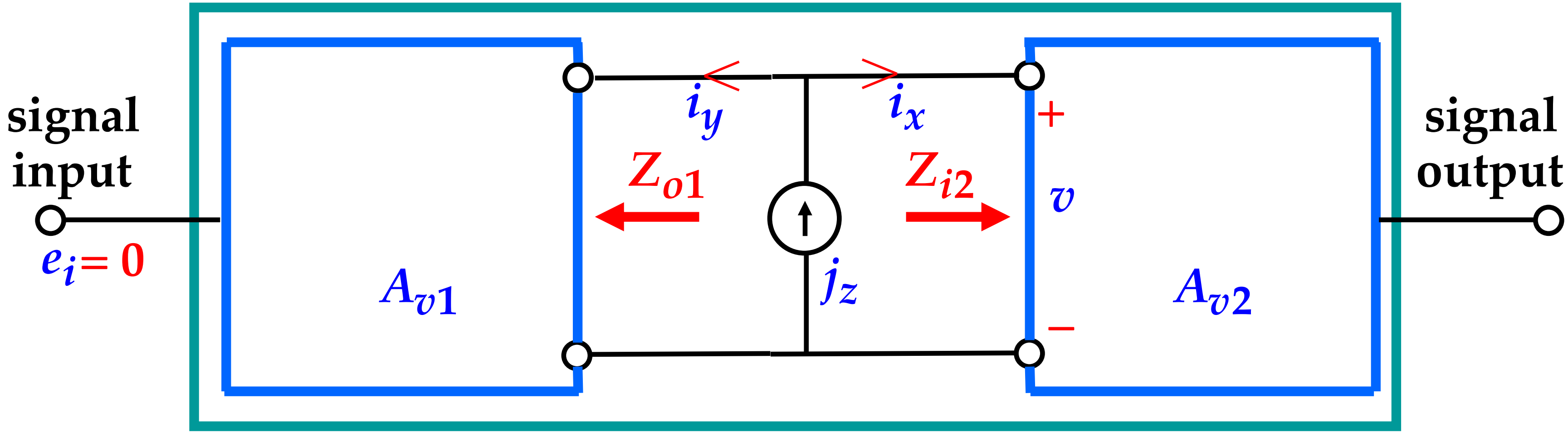
The Chain Theorem (CT)



Nulled i_y means that the A_{v1} box is unloaded, so the input voltage to the A_{v2} box is the open-circuit (oc) output voltage of the A_{v1} box.



Thus, $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$ is the *voltage - buffered gain* of the two stages.



Also

$$T_i \equiv \left.\frac{i_y}{i_x}\right|_{e_i=0} = \left.\frac{v/Z_{o1}}{v/Z_{i2}}\right|_{e_i=0} = \frac{Z_{i2}}{Z_{o1}},$$

so the DT becomes

$$A_{v12} = A_{v1}^{oc} A_{v2} \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

This can be interpreted as

$$\left[\begin{array}{c}\text{gain}\\ \text{of the two stages}\end{array}\right] = \left[\begin{array}{c}\text{voltage buffered gain}\\ \text{of the two stages}\end{array}\right] \text{X} \left[\begin{array}{c}\text{voltage loading factor}\\ \text{between the two stages}\end{array}\right]$$

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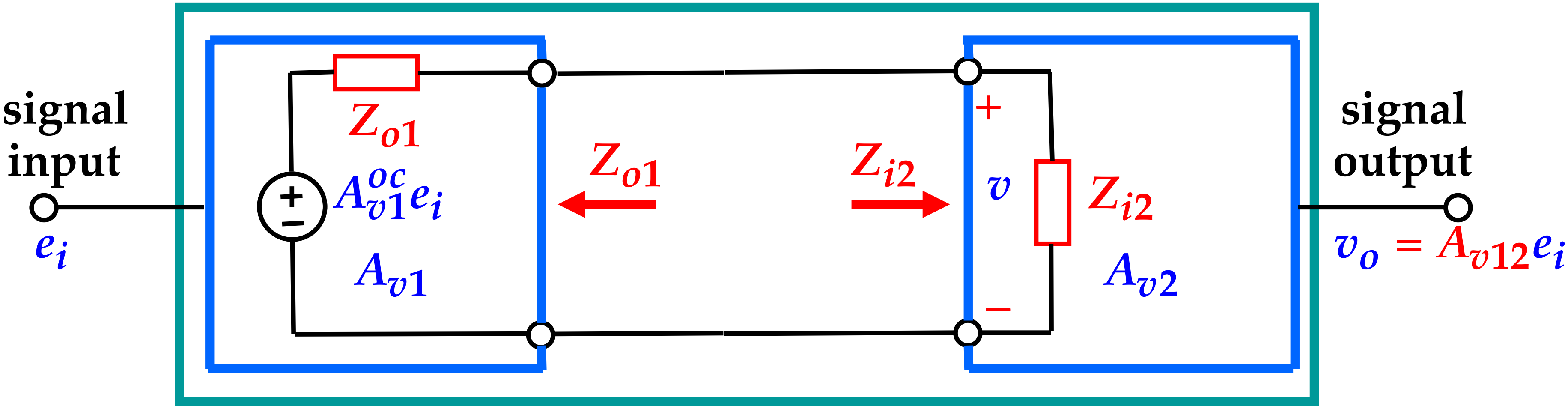
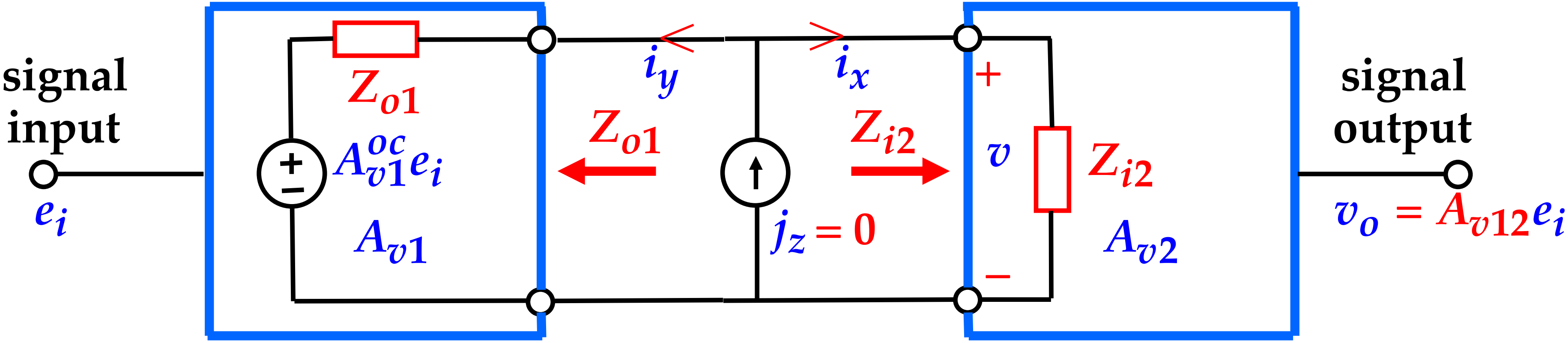
This is exactly the result that would be obtained directly from the model:



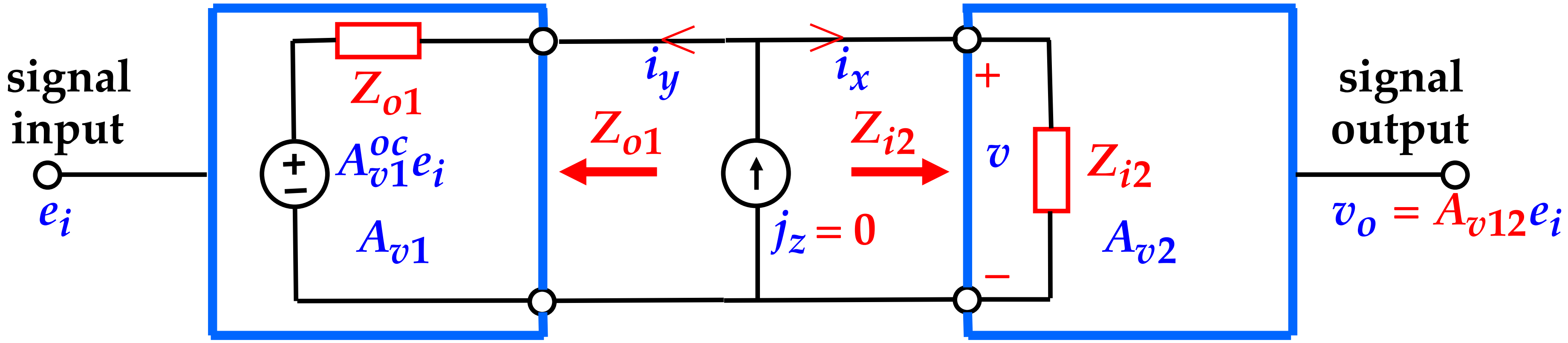
$$A_{v12} = A_{v1}^{oc}A_{v2}\frac{Z_{i2}}{Z_{i2}+Z_{o1}}$$

The Chain Theorem (CT)

A useful application of the DT with $T_{ni} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.

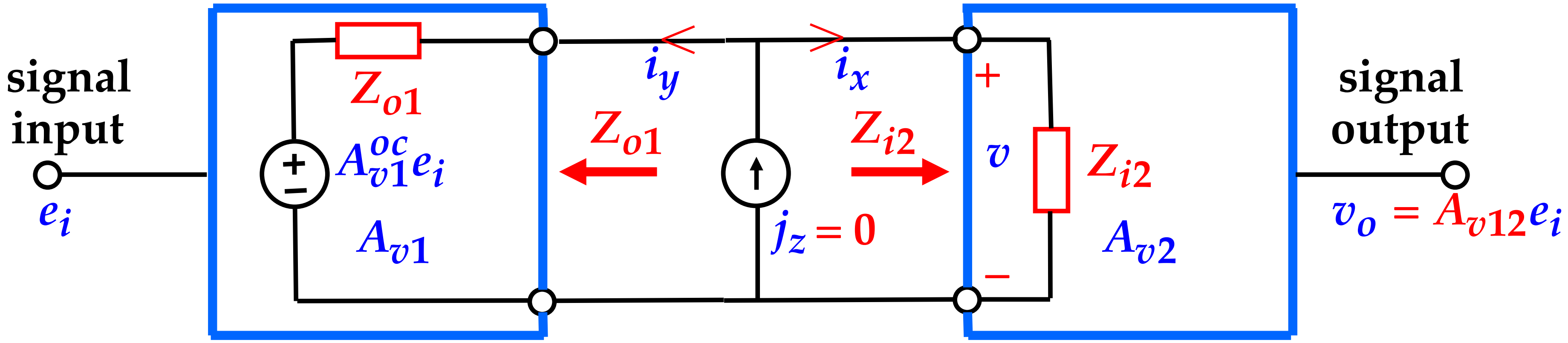


The Chain Theorem (CT)



This "Divide and Conquer" approach avoids analysis of both stages simultaneously.

The Chain Theorem (CT)



$$A_{v12} = A_{v1}^{oc}A_{v2}\frac{1}{1+\frac{1}{T_i}} = A_{v1}^{oc}A_{v2}D_i$$

$$T_i \equiv \frac{Z_{i2}}{Z_{o1}} \qquad D_i \equiv \frac{1}{1+\frac{1}{T_i}} = \frac{T_i}{1+T_i} = \frac{Z_{i2}}{Z_{i2}+Z_{o1}}$$

where $A_{v1}^{oc}A_{v2}$ is the "voltage buffered" gain that would occur if there were a buffer between the two stages, and D_i is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.

The Chain Theorem (CT)

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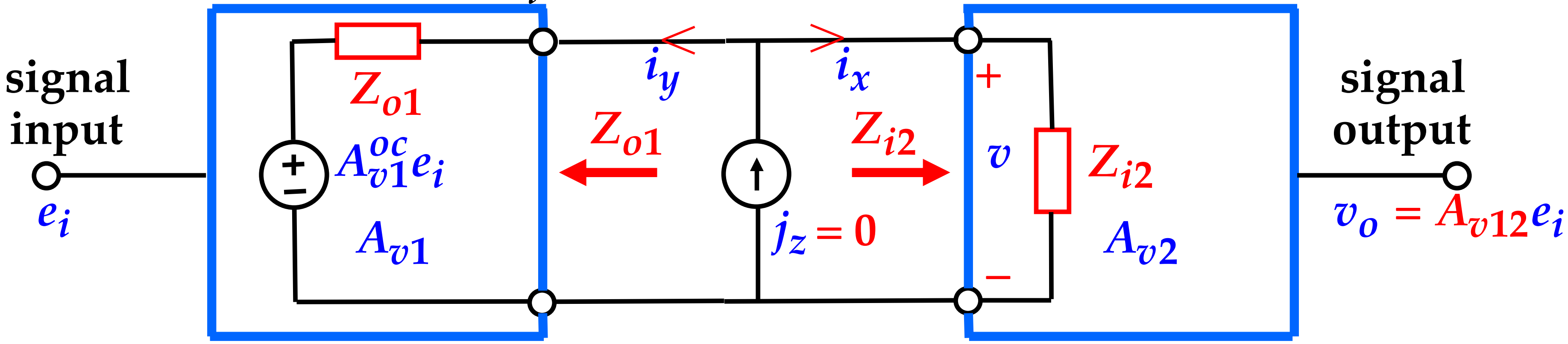


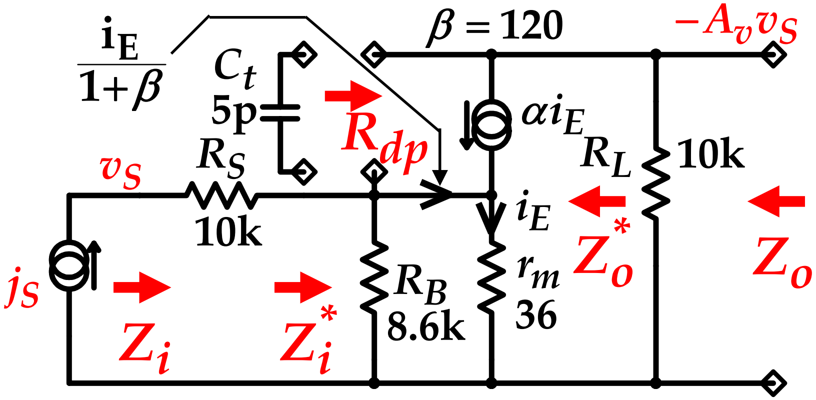
Since all TFs will be in factored pole-zero form, the only place where additional approximation may be needed resides inside the D_i, where the sum of two TFs is required.

"Doing the algebra on the graph" can be conducted in two ways:

$1 + T_i$ can be found as the sum of the TFs 1 and T_i, dominated by the larger;

D_i can be found from $\frac{1}{D_i} = 1 + \frac{1}{T_i} = \frac{1}{1} + \frac{1}{T_i}$ as the reciprocal sum of 1 and T_i, dominated by the smaller.

Let each stage be the 1CE stage previously treated.



$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\,\frac{\alpha R_L}{r_m + \frac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_m = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

$$Z_i = R_{im}\frac{1+sC_tR_{ni}}{1+sC_tR_{di}} \qquad = 82dB\frac{1+\frac{s/2\pi}{51kHz}}{1+\frac{s/2\pi}{39kHz}}$$

$$Z_o = R_{om}\frac{1+sC_tR_{no}}{1+sC_tR_{do}}$$

$$R_{im} \equiv R_S + R_B\|(1+\beta)r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$$R_{om} \equiv R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega$$

$$R_{ni} = mR_L \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L}R_L = 620k$$

$$R_{no} = R_S\|R_B\|(1+\beta)r_m = 2.2k$$

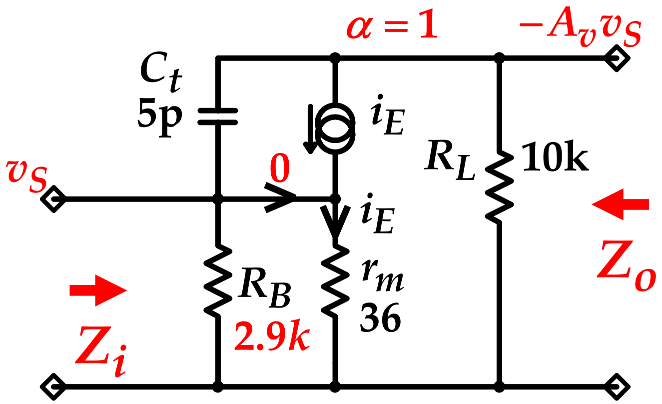
$$R_{di} = \frac{R_B\|(1+\beta)r_m}{R_B\|r_m\|R_L}R_L = 820k$$

$$R_{do} = mR_L = 620k$$

However, to make the symbolic equations more compact, without loss of generality, let $R_S \to 0$ and $\alpha \to 1$ ($\beta \to \infty$).

To keep R_{im}^* the same, also let $R_B \| (1+\beta) r_m = 2.9k \to R_B$

The new 1CE stage is:



$$A_v = A_{vm}\frac{1 - s/\omega_z}{1 + s/\omega_p} = 49dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}}$$

$$A_{vm} = \frac{R_L}{r_m} = 280 \Rightarrow 49dB$$

$$R_n = r_m = 36\Omega \qquad R_d = R_L = 10k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m = 1$$

$$Z_i = R_{im}\frac{1 + sC_tR_L}{1 + sC_tR_L\frac{R_B}{R_B \| r_m \| R_L}} = 69dB\frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

$$Z_o = R_{om}\frac{1}{1 + sC_tR_L} = 80dB\frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$$

$R_{im} = R_B = 2.9k \Rightarrow 69dB$ ref. 1Ω

$R_{om} = R_L = 10k \Rightarrow 80dB$ ref. 1Ω

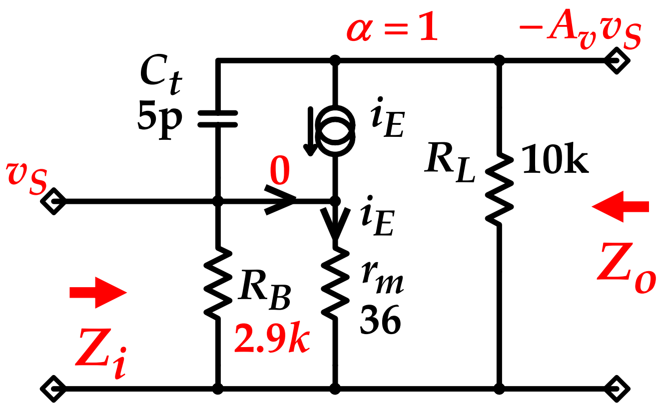
$R_{ni} = mR_L = 10k$

$R_{no} = 0$

$$R_{di} = \frac{R_B \| (1+\beta)r_m}{R_B \| r_m \| R_L}R_L = 820k$$

$R_{do} = mR_L = 10k$

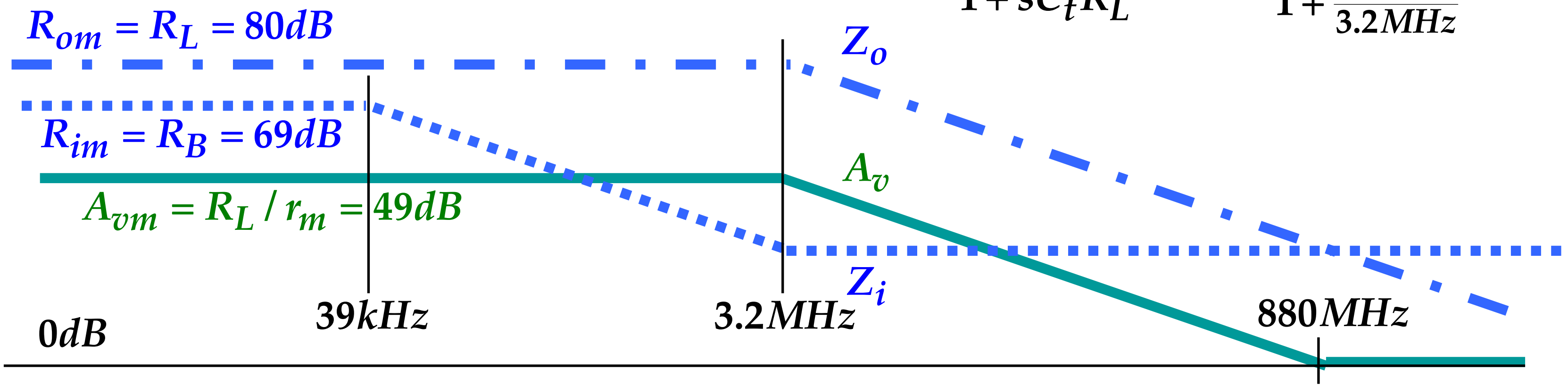
Note that letting $R_S \to 0$ reduces the "miller multiplier" m to 1.

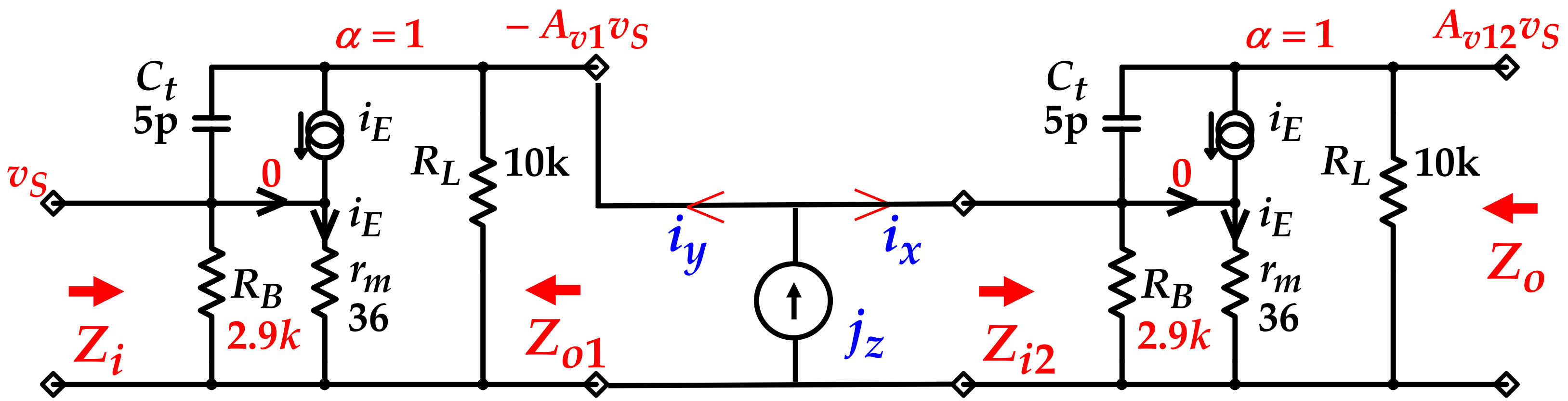


$$A_v = \frac{R_L}{r_m}\frac{1 - sC_t r_m}{1 + sC_t R_L} = 49dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}}$$

$$Z_i = R_B\frac{1 + sC_t R_L}{1 + sC_t R_L\frac{R_B}{R_B \| r_m \| R_L}} = 69dB\frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

$$Z_o = R_L\frac{1}{1 + sC_t R_L} = 80dB\frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$$



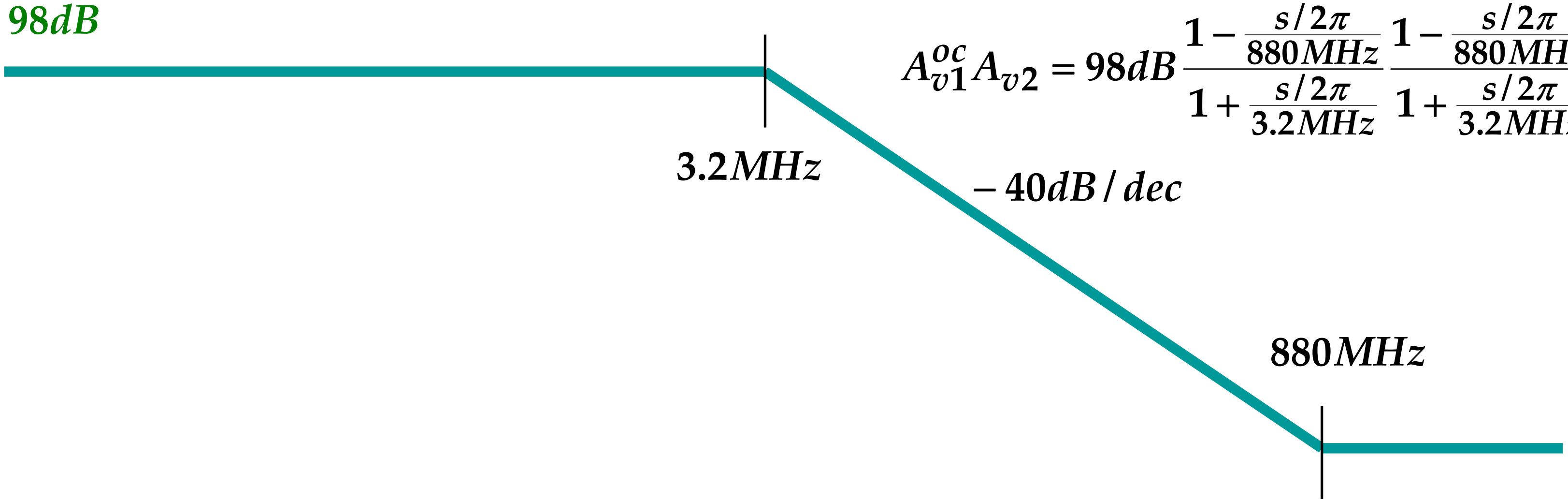


The DT gives

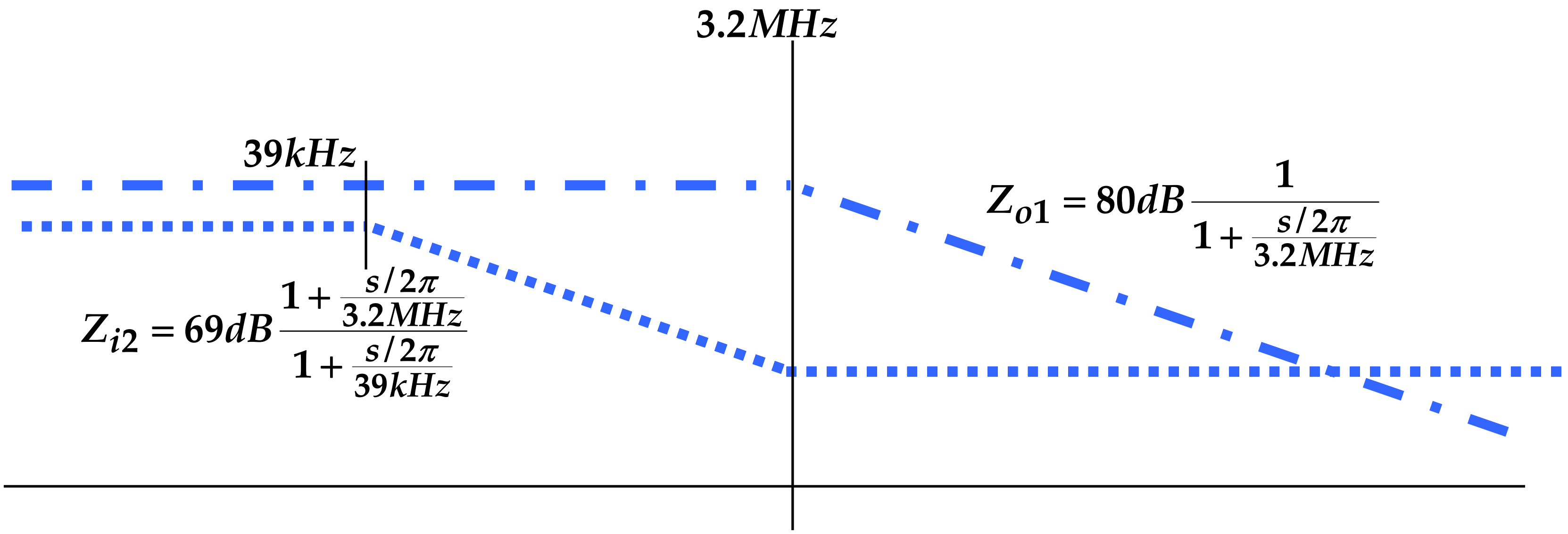
$$A_{v12} = A_{v1}^{oc} A_{v2} D_i \qquad (T_{ni} = \infty)$$

The buffered gain $A_{v1}^{oc} A_{v2}$ is the product of the two separate gains, where A_{v1} is already open-circuit:

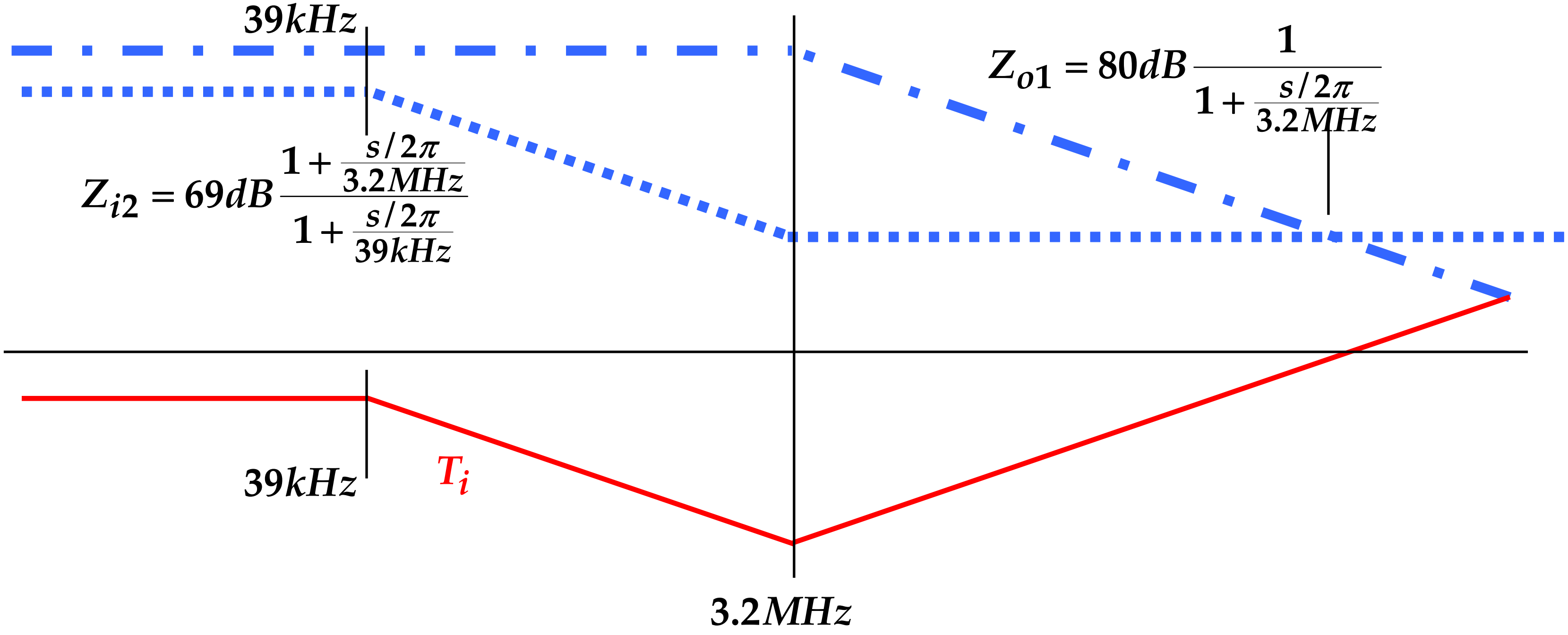
$$A_{v1}^{oc} A_{v2} = 98dB \left(\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}} \right)^2$$



$$A_{v1}^{oc} A_{v2} = 98dB \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}} \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}}$$


$3.2MHz$
$39kHz$
$Z_{o1} = 80dB\frac{1}{1+\frac{s/2\pi}{3.2MHz}}$
$Z_{i2} = 69dB\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{39kHz}}$

$$T_i \equiv \frac{Z_{i2}}{Z_{o1}} = -11dB \frac{\left(1+\frac{s/2\pi}{3.2MHz}\right)^2}{1+\frac{s/2\pi}{39kHz}}$$

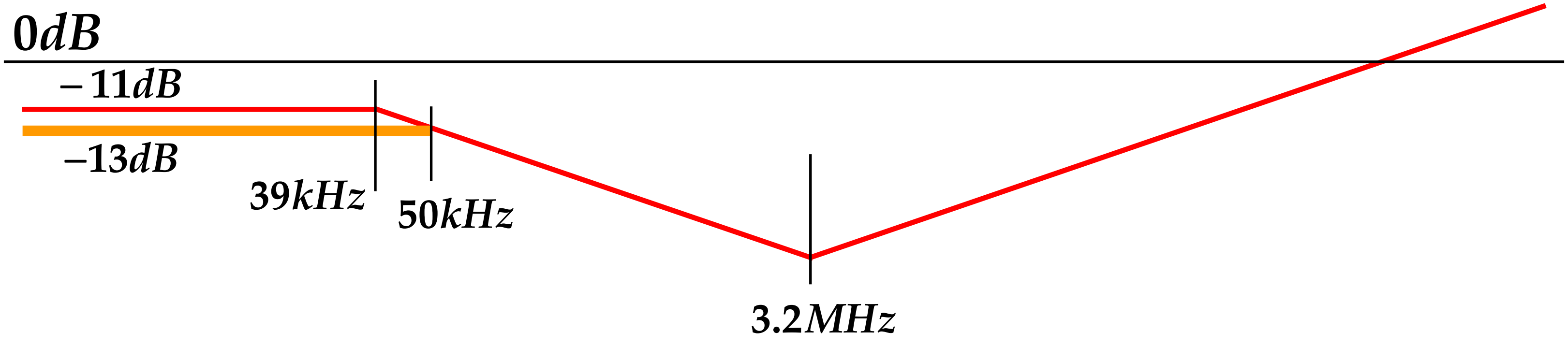


$$Z_{o1} = 80dB \frac{1}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Z_{i2} = 69dB \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{39kHz}}$$

The discrepancy factor $D_i = \dfrac{1}{1+\frac{1}{T_i}}$ **or** $\dfrac{1}{D_i} = \dfrac{1}{1} + \dfrac{1}{T_i}$ **or** $D_i = 1 \| T_i$

is dominated by the smaller:



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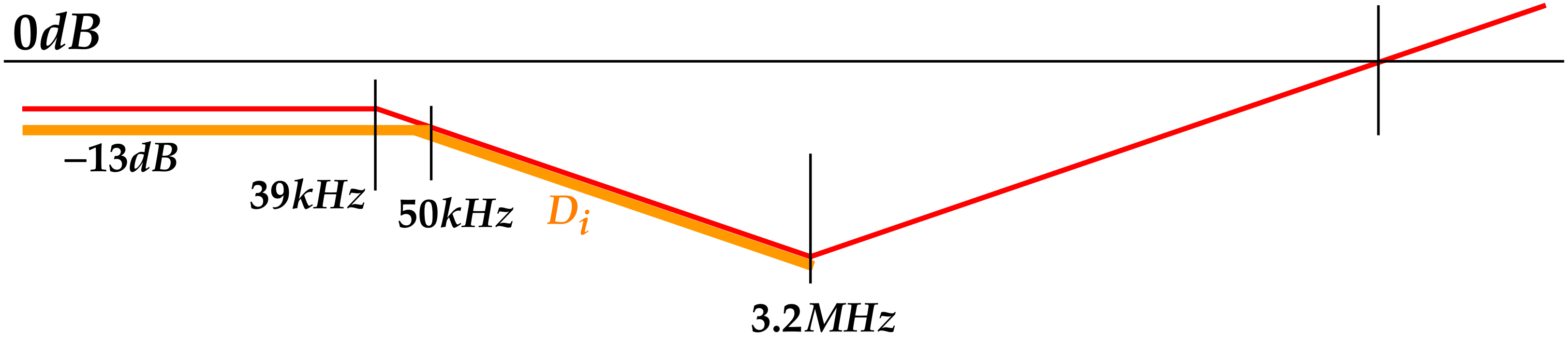

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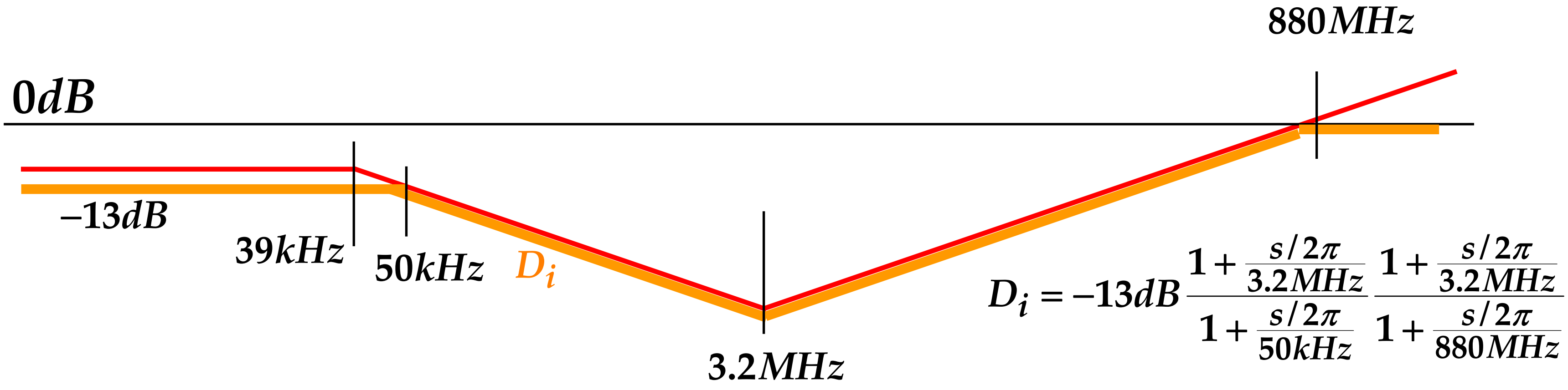
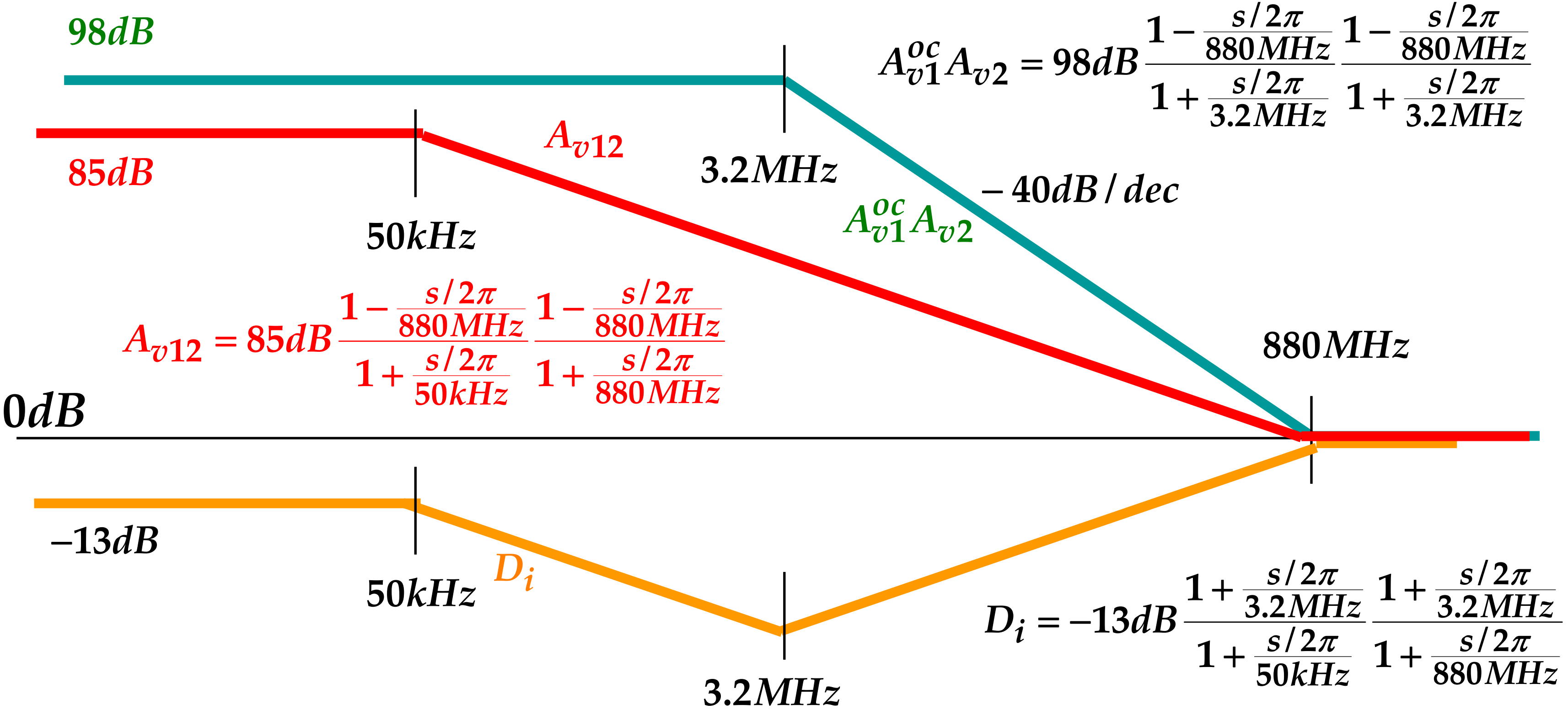


$$D_i = -13dB \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{50kHz}} \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{880MHz}}$$

All these graphical constructions can be conducted symbolically to give the result for D_i in low entropy factored pole-zero form.

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:

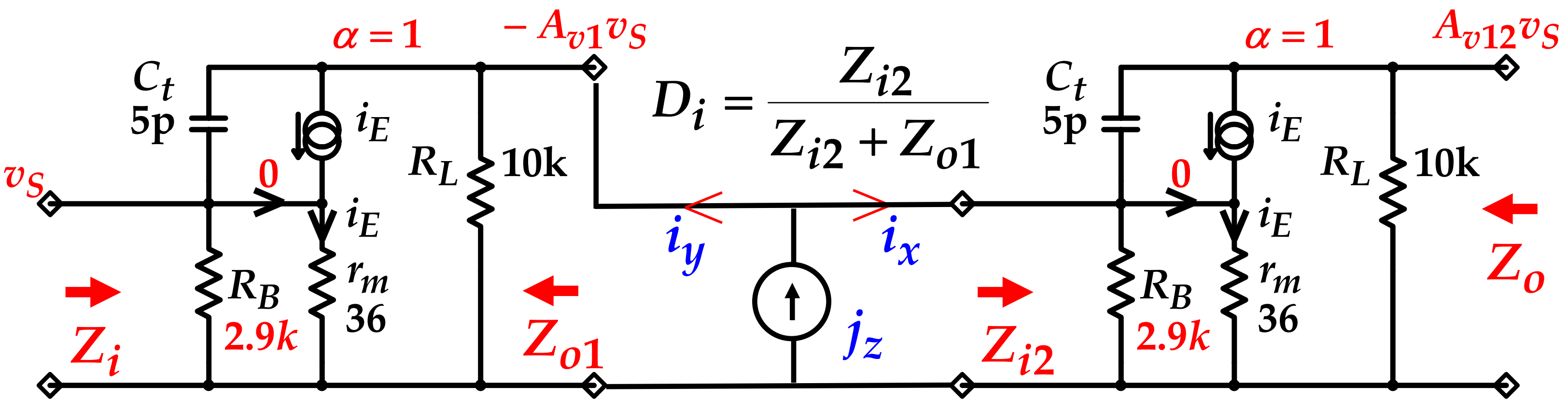


$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

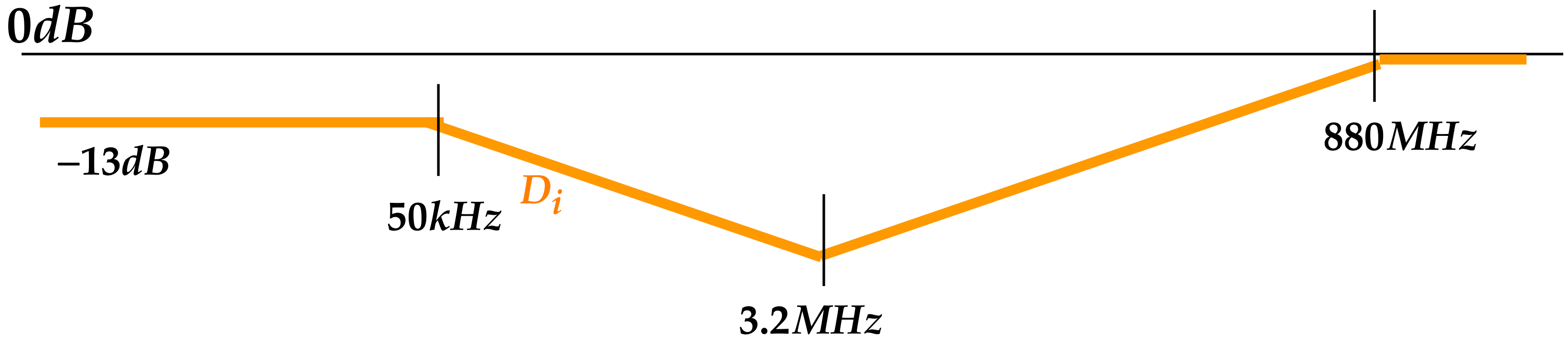
$$A_{v12} = 85dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

$$D_i = -13dB\,\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{50kHz}}\,\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{880MHz}}$$

The fact that D_i is less than 1 over most of the frequency range indicates that the second stage imposes heavy loading upon the first stage:



$$D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$



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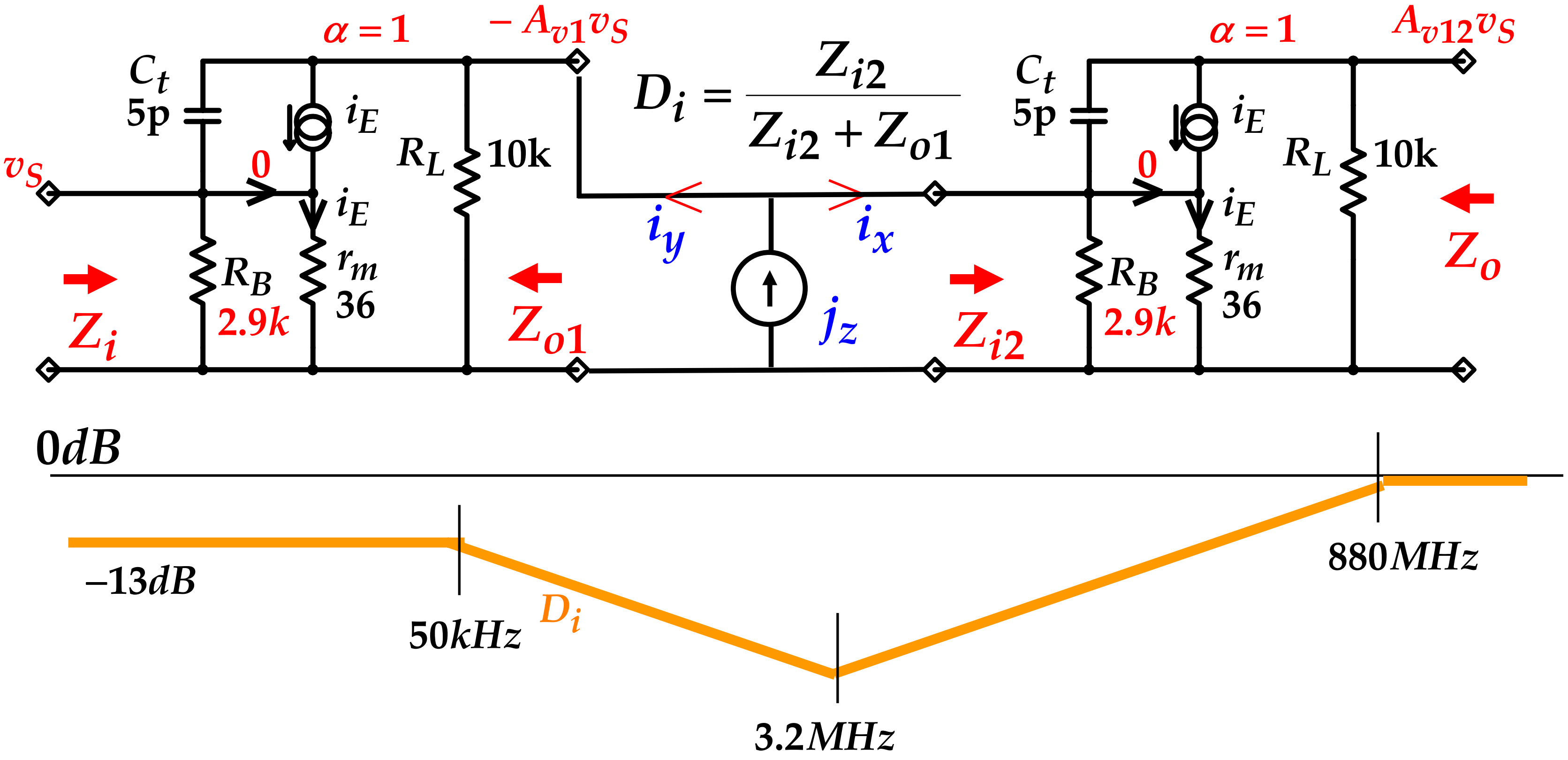
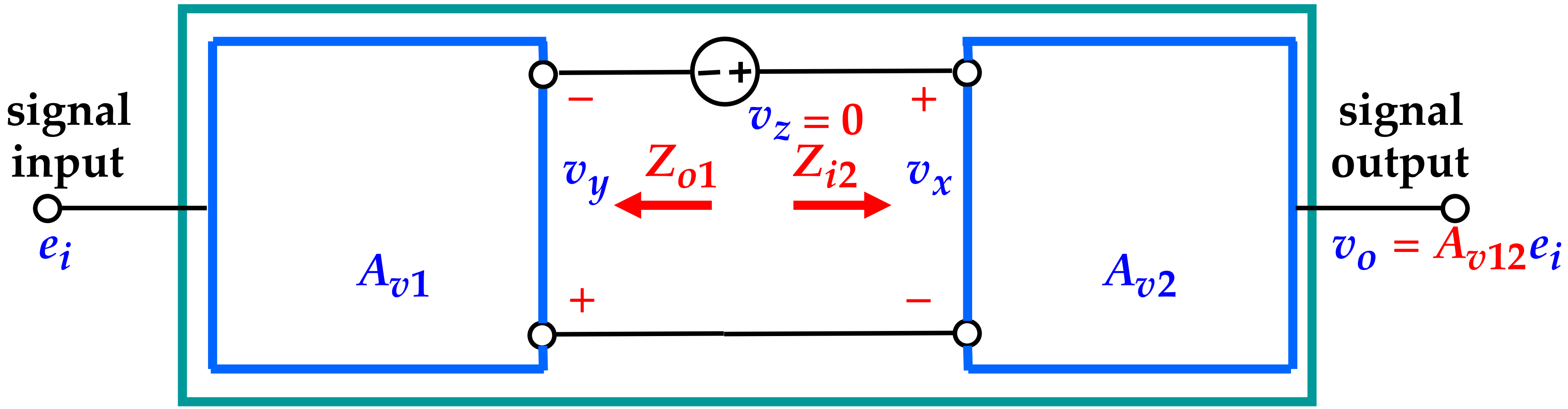

$$D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

This suggests that the first stage behaves more like a current source than a voltage source, and therefore that the analysis might be better undertaken using the dual form of the DT.

The Chain Theorem (CT)



The gain $A_{v12} \equiv \frac{v_o}{e_i}$ is given by the DT:

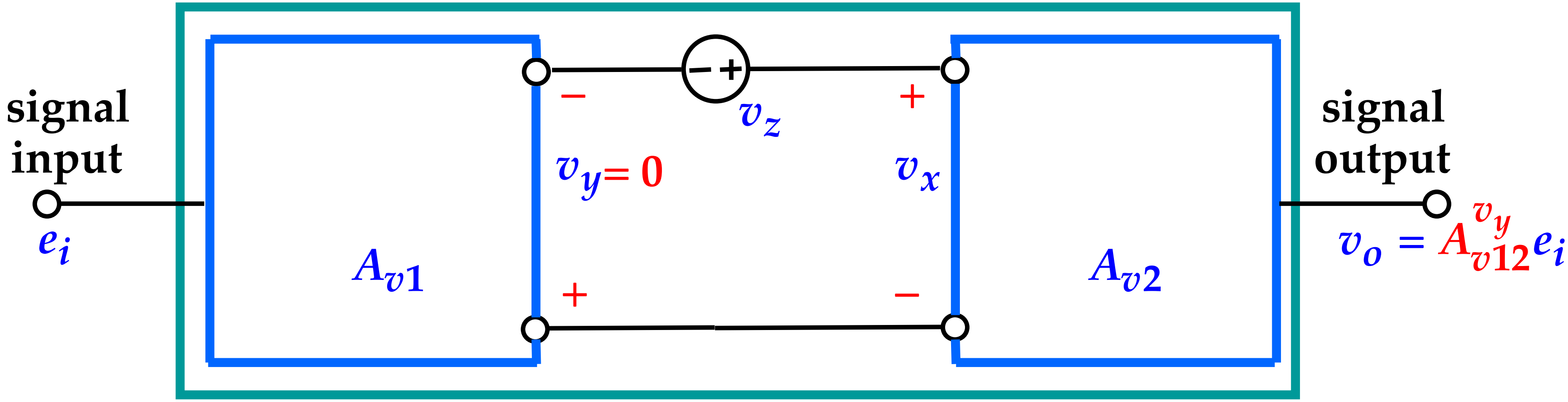
$$A_{v12} = A_{v12}^{v_y}\frac{1+\frac{1}{T_{nv}}}{1+\frac{1}{T_v}}$$

The TF $T_{nv} \equiv v_y/v_x\big|_{v_o=0}$ is an ndi calculation with the output v_o nulled.

If v_o is nulled, so is v_x, so $T_{nv} = \infty$.

This implies that T_{nv} is infinite unless the signal can bypass the injection point.

The Chain Theorem (CT)



signal input

e_i

A_{v1}

$-$ v_z $+$

$v_y = 0$

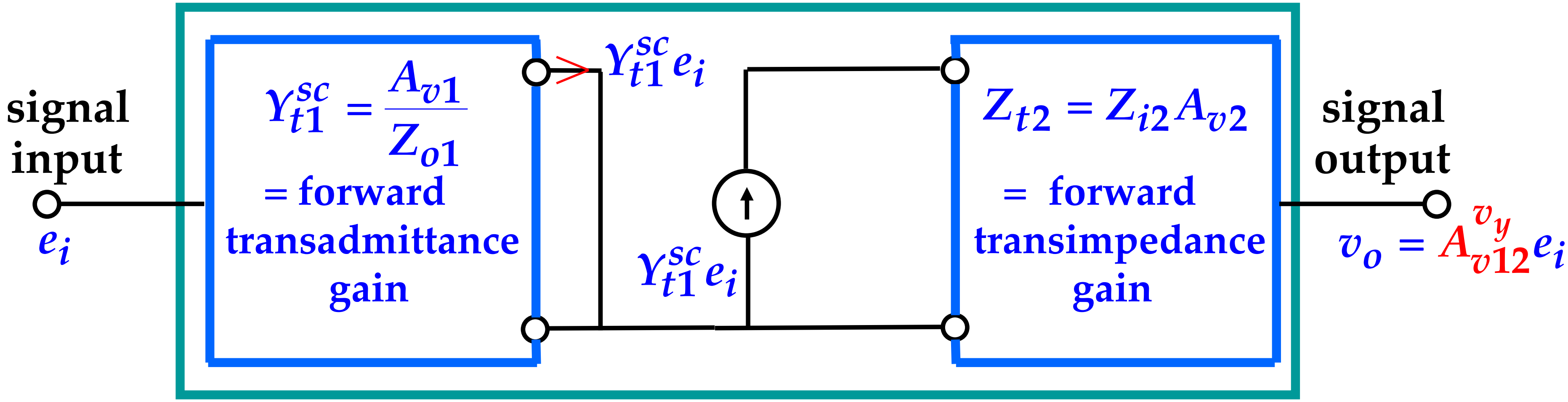
$+$ v_x $-$

A_{v2}

signal output

$v_o = A_{v12}^{v_y} e_i$

Nulled v_y means that the A_{v1} box is shorted, so the input current to the A_{v2} box is the short-circuit (sc) output current of the A_{v1} box.



signal input

e_i

$Y_{t1}^{sc} = \frac{A_{v1}}{Z_{o1}}$

= forward transadmittance gain

$Y_{t1}^{sc} e_i$

$Y_{t1}^{sc} e_i$

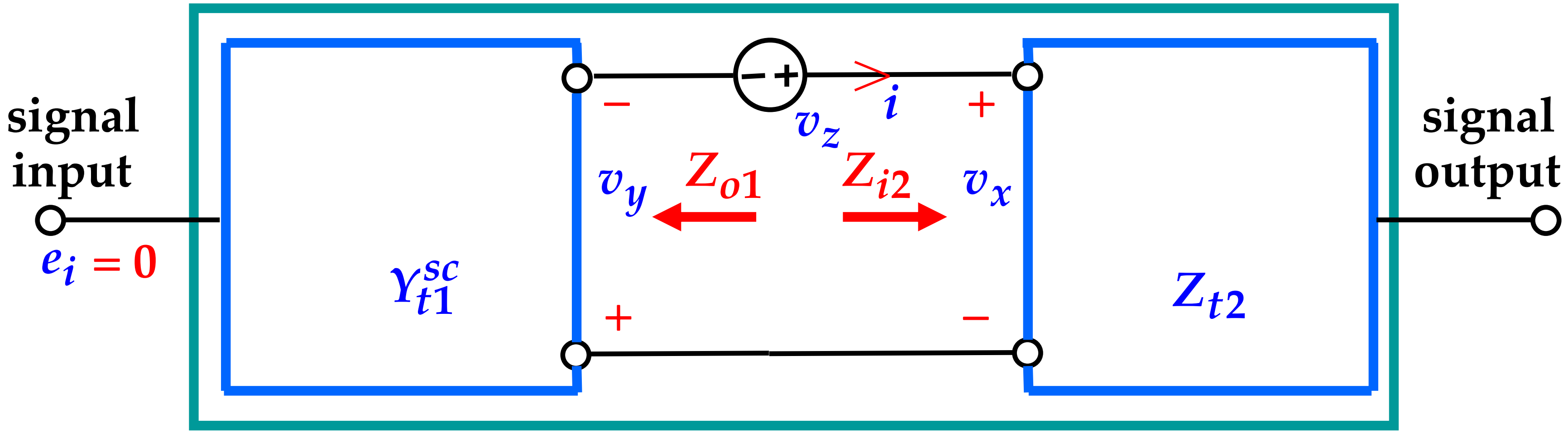
$Z_{t2} = Z_{i2} A_{v2}$

= forward transimpedance gain

signal output

$v_o = A_{v12}^{v_y} e_i$

Thus, $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$ is the *current - buffered gain* of the two stages.



Also

$$T_v \equiv \left.\frac{v_y}{v_x}\right|_{e_i=0} = \left.\frac{iZ_{o1}}{iZ_{i2}}\right|_{e_i=0} = \frac{Z_{o1}}{Z_{i2}},$$

so the DT becomes

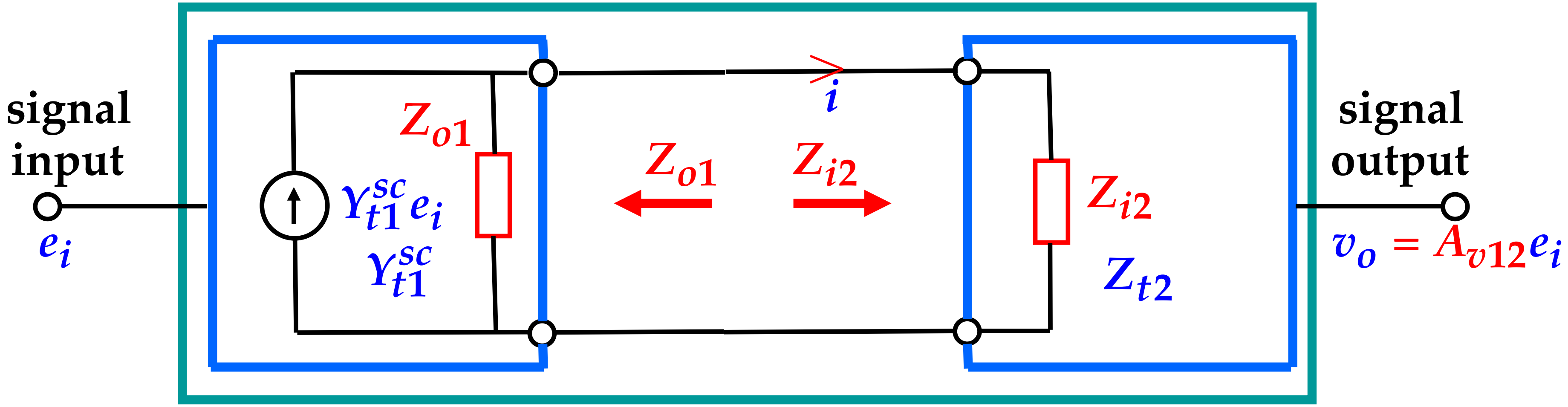
$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

This can be interpreted as

$$\left[\begin{matrix}\text{gain} \\ \text{of the two stages}\end{matrix}\right] = \left[\begin{matrix}\text{current buffered gain} \\ \text{of the two stages}\end{matrix}\right] \text{X} \left[\begin{matrix}\text{current loading factor} \\ \text{between the two stages}\end{matrix}\right]$$

The Chain Theorem (CT)

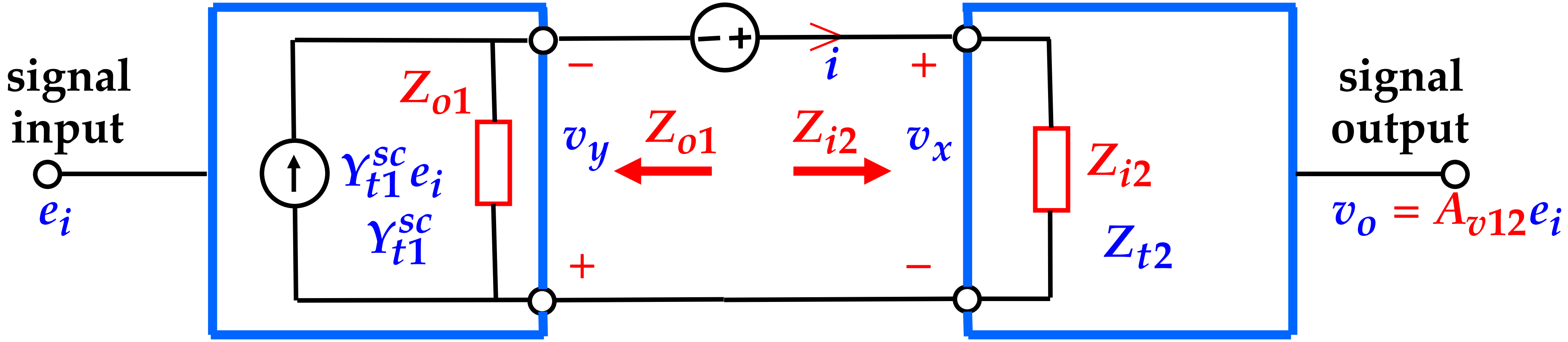
This is exactly the result that would be obtained directly from the model:



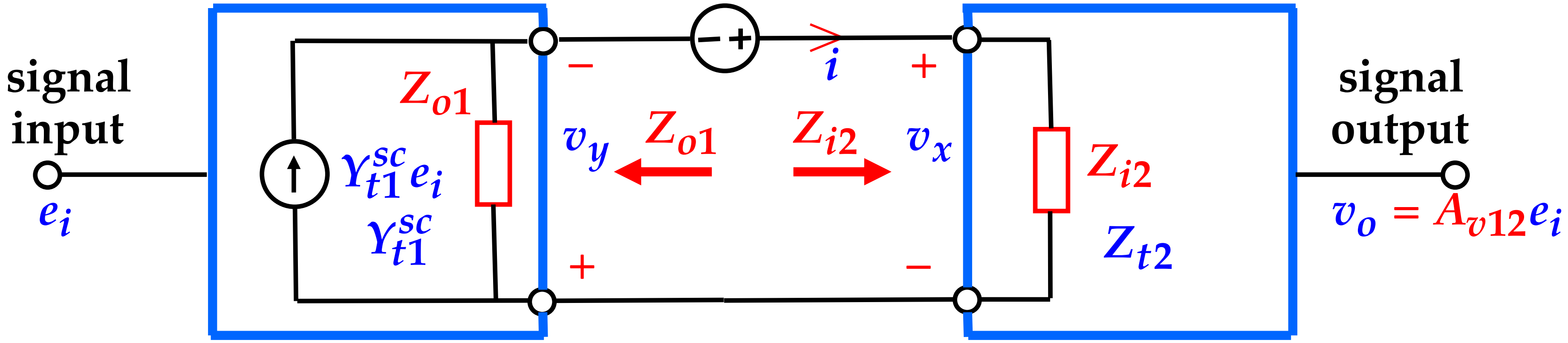
$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

The Chain Theorem (CT)

A useful application of the DT with $T_{nv} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.

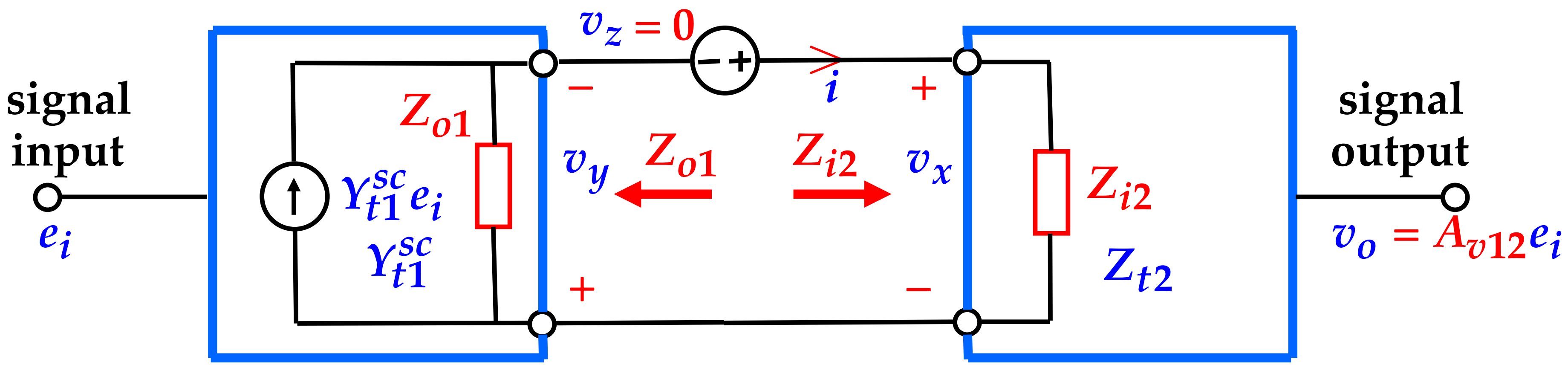


The Chain Theorem (CT)



This "Divide and Conquer" approach avoids analysis of both stages simultaneously.

The Chain Theorem (CT)



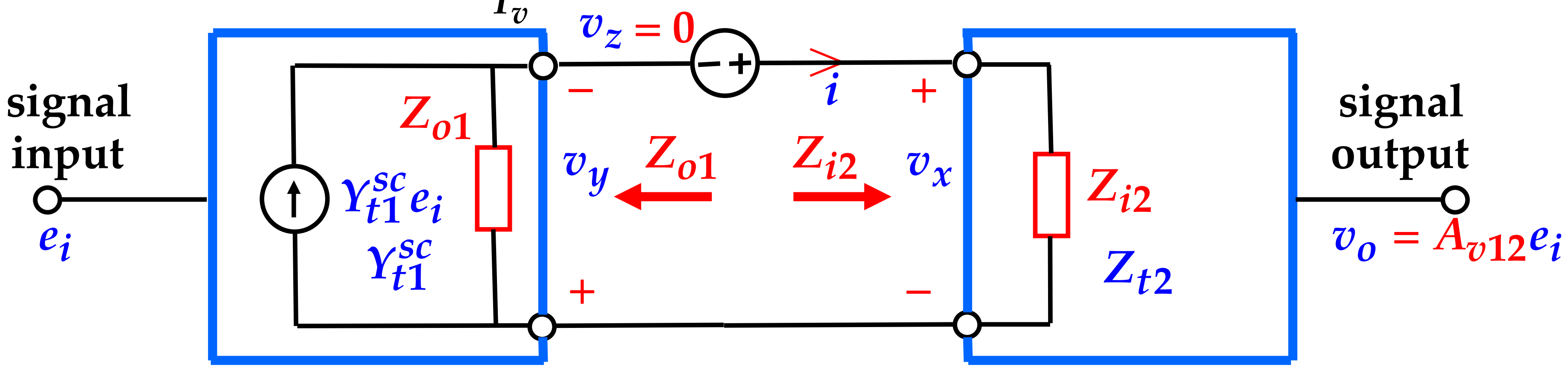
$$A_{v12} = Y_{t1}^{sc}Z_{t2}\frac{1}{1+\frac{1}{T_v}} = Y_{t1}^{sc}Z_{t2}D_v$$

$$T_v \equiv \frac{Z_{o1}}{Z_{i2}} \qquad D_v \equiv \frac{1}{1+\frac{1}{T_v}} = \frac{T_v}{1+T_v} = \frac{Z_{o1}}{Z_{i2}+Z_{o1}}$$

where $Y_{t1}^{sc}Z_{t2}$ is the "current buffered" gain that would occur if there were a buffer between the two stages, and D_v is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.

The Chain Theorem (CT)

$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{1}{1+\frac{1}{T_v}} = Y_{t1}^{sc} Z_{t2} D_v$$



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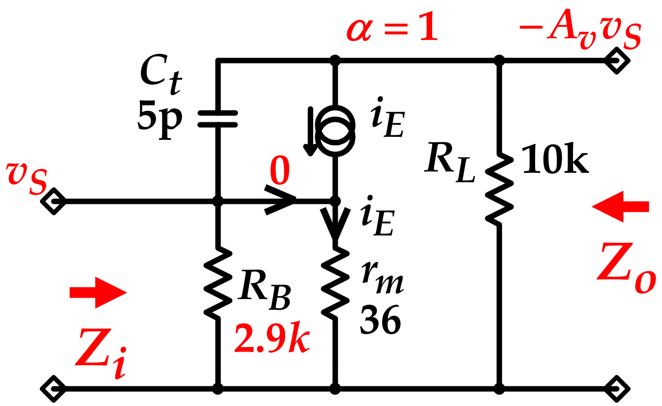
$1 + T_v$ can be found as the sum of the TFs 1 and T_v, dominated by the larger;

D_v can be found from $\frac{1}{D_v} = 1 + \frac{1}{T_v} = \frac{1}{1} + \frac{1}{T_v}$ as the reciprocal sum of 1 and T_v dominated by the smaller.

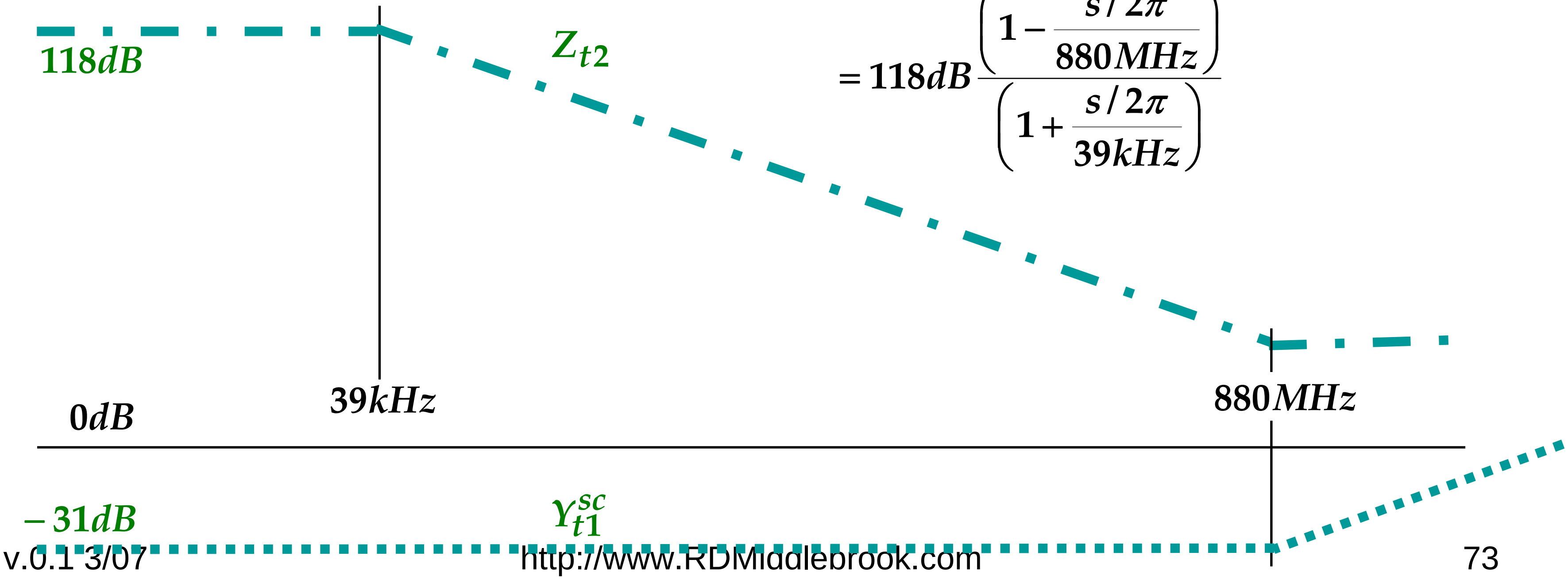


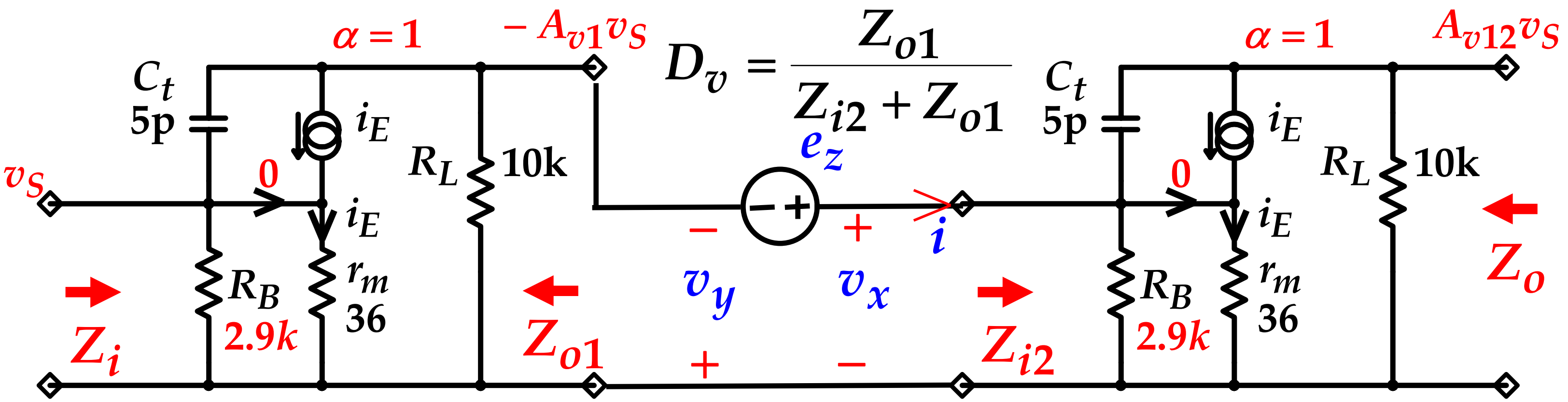
$$Y_t^{sc} = \frac{A_v}{Z_o} = \frac{1}{r_m}\left(1 - sC_t r_m\right)$$

$$= -31dB\left(1 - \frac{s/2\pi}{880MHz}\right)$$

$$Z_t = Z_i A_v = \frac{R_B R_L}{r_m}\frac{\left(1 - sC_t r_m\right)}{\left(1 + sC_t R_L \frac{R_B}{R_B \| r_m \| R_L}\right)}$$

$$= 118dB\frac{\left(1 - \frac{s/2\pi}{880MHz}\right)}{\left(1 + \frac{s/2\pi}{39kHz}\right)}$$





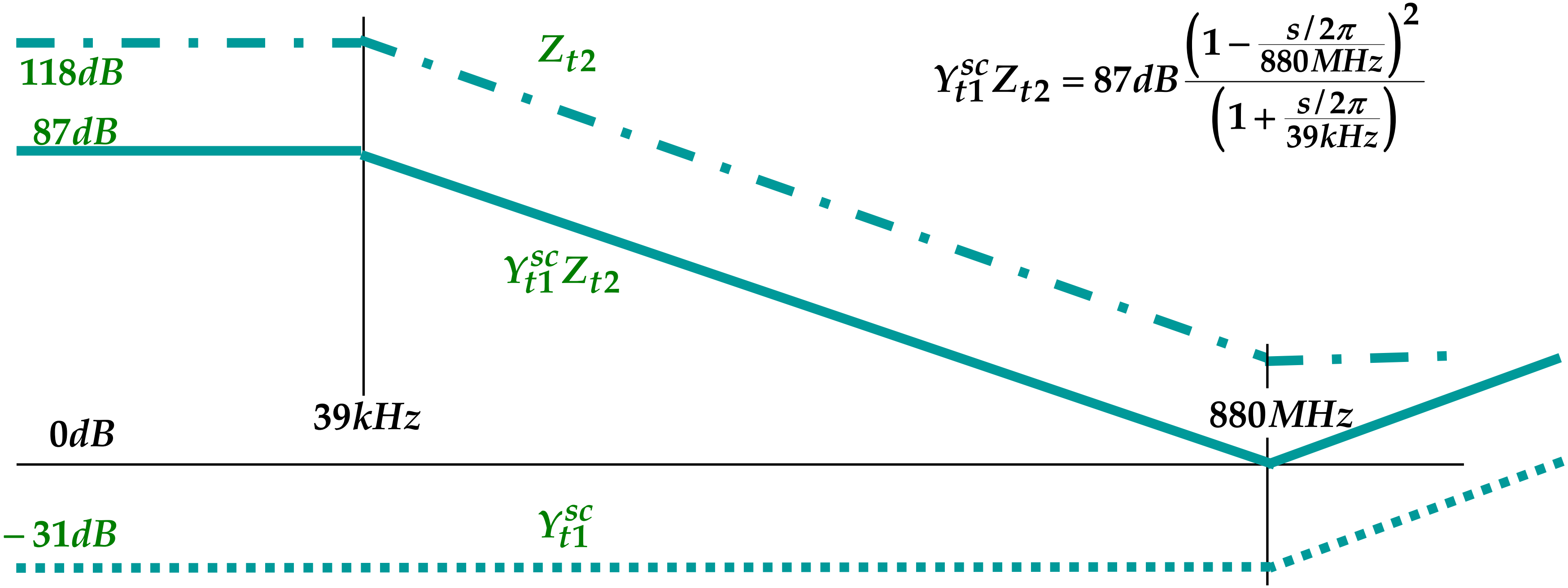
$$D_v = \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

The DT gives

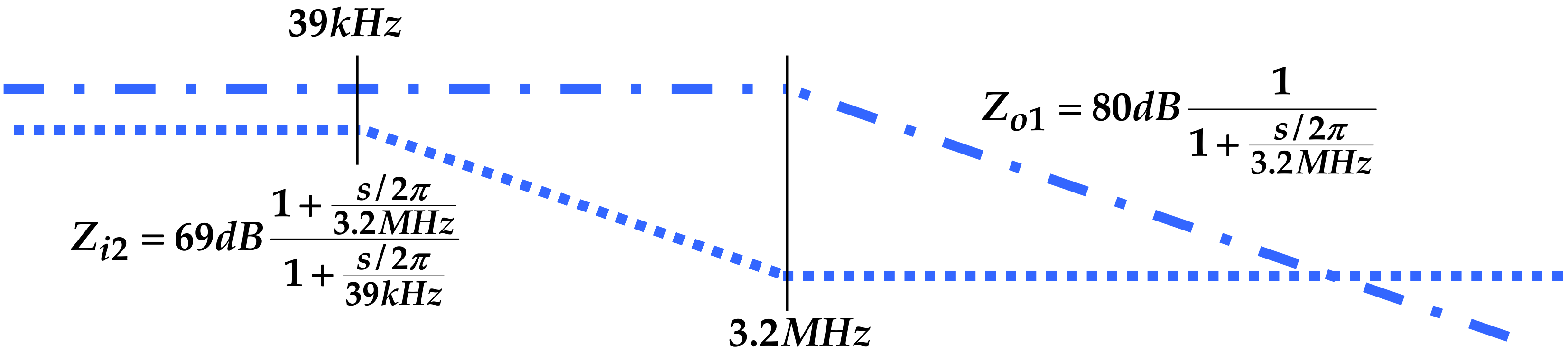
$$A_{v12} = Y_{t1}^{sc} Z_{t2} D_v \qquad (T_{nv} = \infty)$$

The current buffered gain $Y_{t1}^{sc} Z_{t2}$ is the product of the two separate gains:

$$Y_{t1}^{sc} Z_{t2} = 87dB \frac{\left(1 - \frac{s/2\pi}{880MHz}\right)^2}{\left(1 + \frac{s/2\pi}{39kHz}\right)}$$



$$Y_{t1}^{sc}Z_{t2} = 87dB\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$

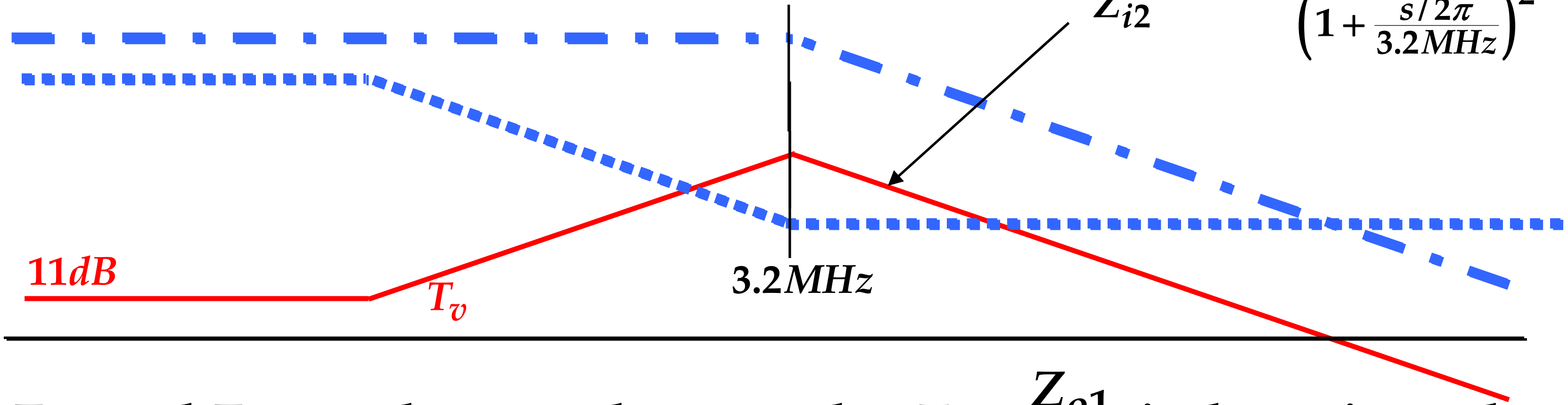


$$Z_{o1} = 80dB\frac{1}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Z_{i2} = 69dB\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{39kHz}}$$

Z_{o1} and Z_{i2} are the same, but note that $T_v = \frac{Z_{o1}}{Z_{i2}}$ is the reciprocal of $T_i = \frac{Z_{i2}}{Z_{o1}}$:

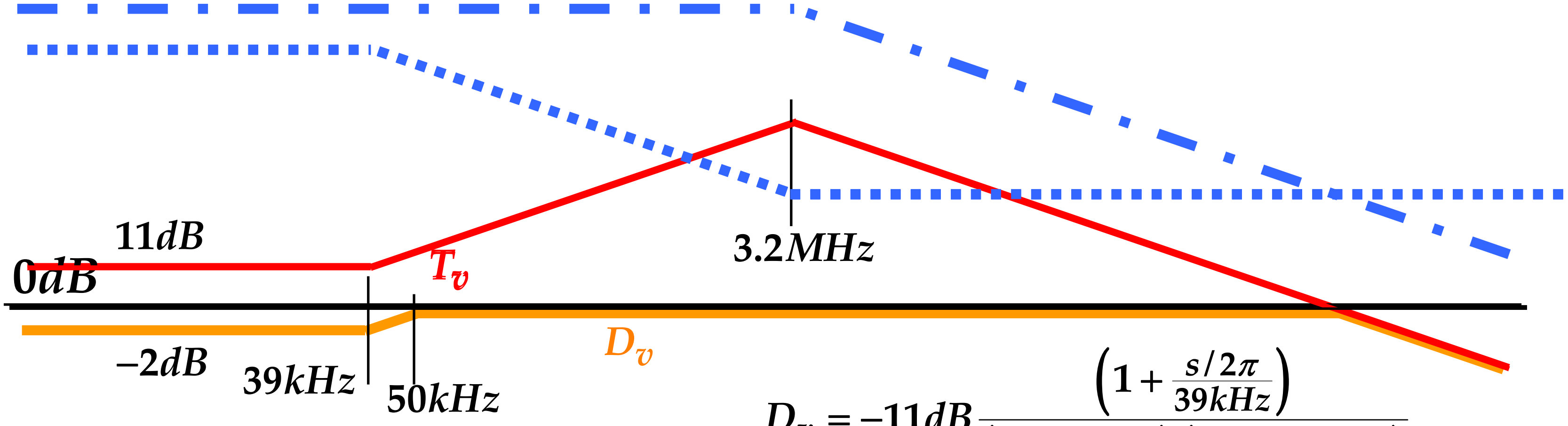
$$T_v = \frac{Z_{o1}}{Z_{i2}} = 11dB \frac{\left(1 + \frac{s/2\pi}{39kHz}\right)}{\left(1 + \frac{s/2\pi}{3.2MHz}\right)^2}$$



Z_{o1} and Z_{i2} are the same, but note that $T_v = \frac{Z_{o1}}{Z_{i2}}$ is the reciprocal of $T_i = \frac{Z_{i2}}{Z_{o1}}$:

The discrepancy factor $D_v = \frac{1}{1+\frac{1}{T_v}}$ or $\frac{1}{D_v} = \frac{1}{1} + \frac{1}{T_v}$ or $D_v = 1 \| T_v$

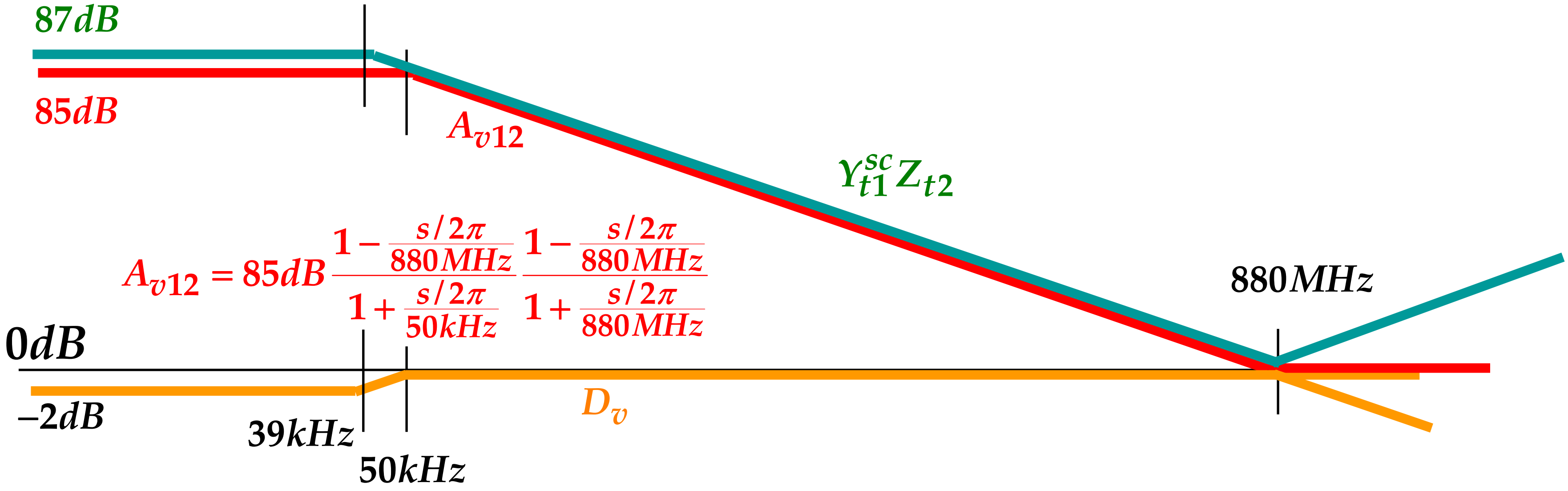
is dominated by the smaller:



$$D_v = -11dB \frac{\left(1+\frac{s/2\pi}{39kHz}\right)}{\left(1+\frac{s/2\pi}{50kHz}\right)\left(1+\frac{s/2\pi}{880MHz}\right)}$$

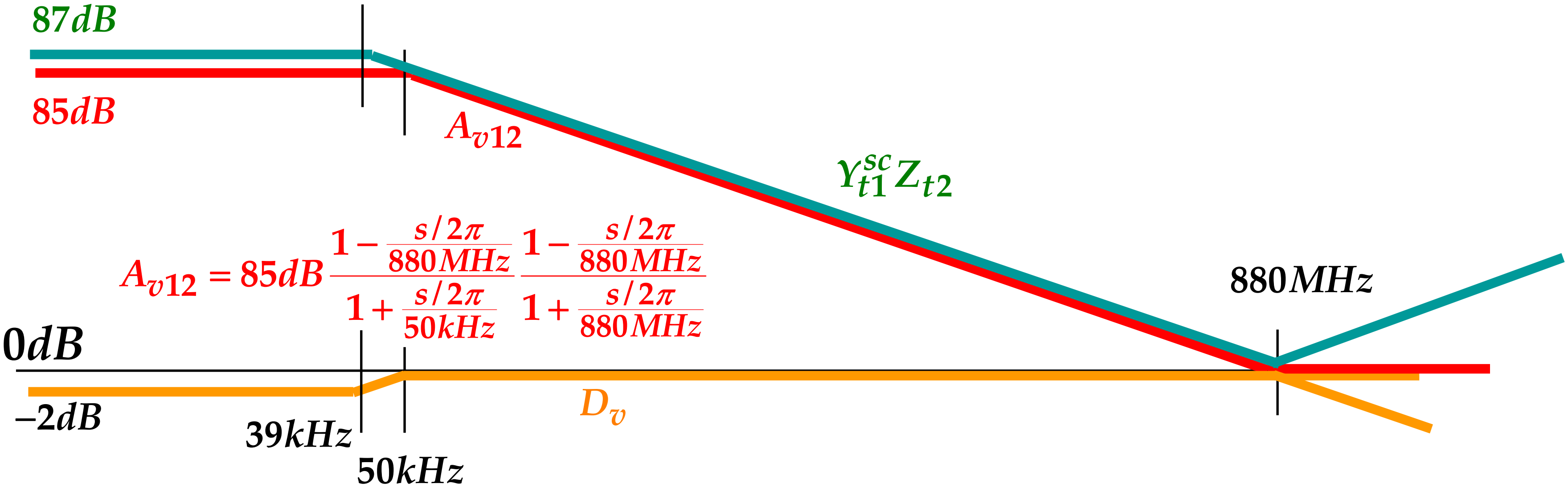
All these graphical constructions can be conducted symbolically to give the result for D_v in low entropy factored pole-zero form.

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:



$$A_{v12} = 85dB \frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}} \frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:



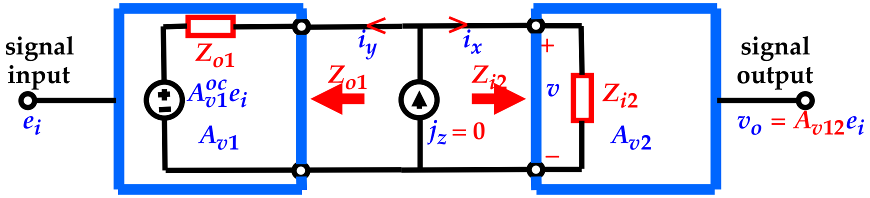
$$A_{v12} = 85dB \frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}} \frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

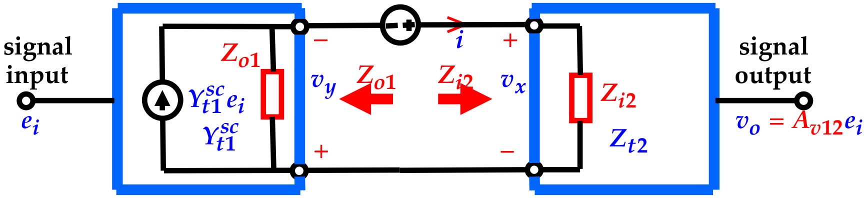
The fact that D_v is close to 1 over most of the frequency range confirms the expectation that the first stage behaves more like a current source than a voltage source.

Summary:

The DT allows assembly of the properties of a 2-stage amplifier from the properties of each separate stage.

This can be done by injection of either a test current j_z or a test voltage e_z at the interface:





$$A_{v12} = A_{v12}^{i_y} D_i$$

where $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$

is the voltage buffered gain

and $D_i = \dfrac{Z_{i2}}{Z_{i2} + Z_{o1}}$

$$A_{v12} = A_{v12}^{v_y} D_v$$

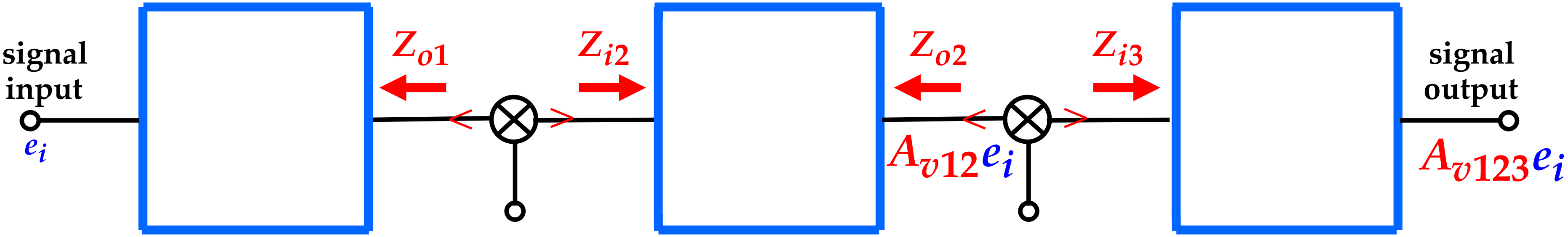
where $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$

is the current buffered gain

and $D_v = \dfrac{Z_{o1}}{Z_{i2} + Z_{o1}}$

are the discrepancy factors representing the interface loading.

In principle, this procedure can be extended to the addition of extra stages:



In practice, this procedure becomes cumbersome because the discrepancy factor for the first interface changes when a second interface is added.

However, there is an alternative form for the gain of 2 stages that circumvents this problem.

The DT results already obtained are:

$$A_{v12} = A_{v12}^{i_y} D_i = A_{v12}^{i_y} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \qquad A_{v12} = A_{v12}^{v_y} D_v = A_{v12}^{v_y} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

Rewrite:

$$\frac{1}{A_{v12}} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}^{i_y}} \qquad \frac{1}{A_{v12}} \frac{Z_{o1}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}^{v_y}}$$

Add the two:

$$\frac{1}{A_{v12}} = \frac{1}{A_{v12}^{i_y}} + \frac{1}{A_{v12}^{v_y}}$$

$$\frac{1}{A_{v12}} = \frac{1}{A_{v12}^{i_y}} + \frac{1}{A_{v12}^{v_y}}$$

This simple and elegant result says that the interface discrepancy factors D_i and D_v are not needed, and the overall gain is a "parallel combination" of the two buffered gains:

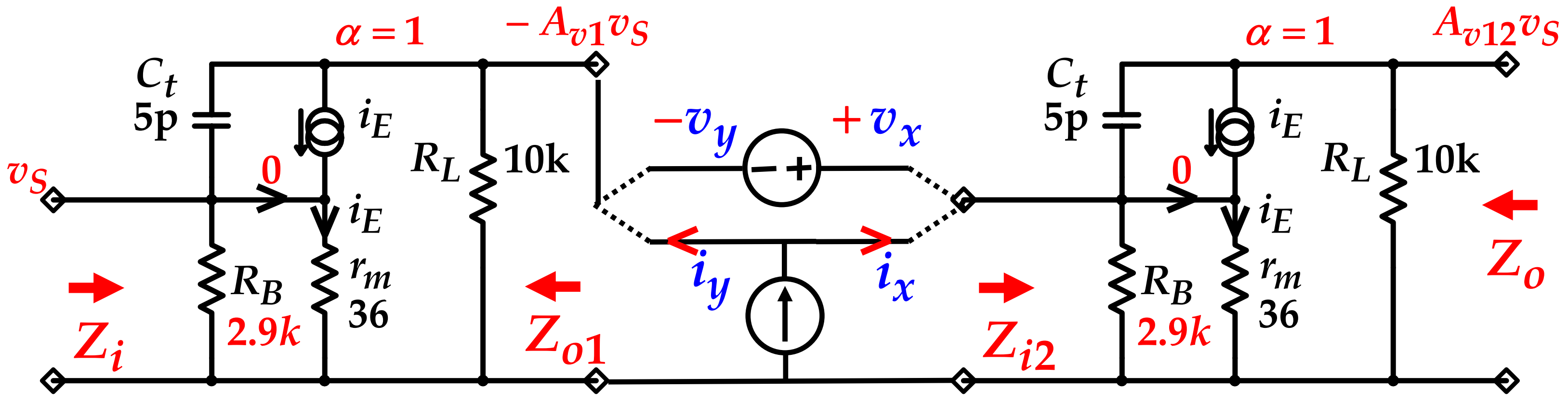
$$A_{v12} = A_{v12}^{i_y} \Big\| A_{v12}^{v_y}$$

where $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$ = **voltage buffered gain of the 2 stages**

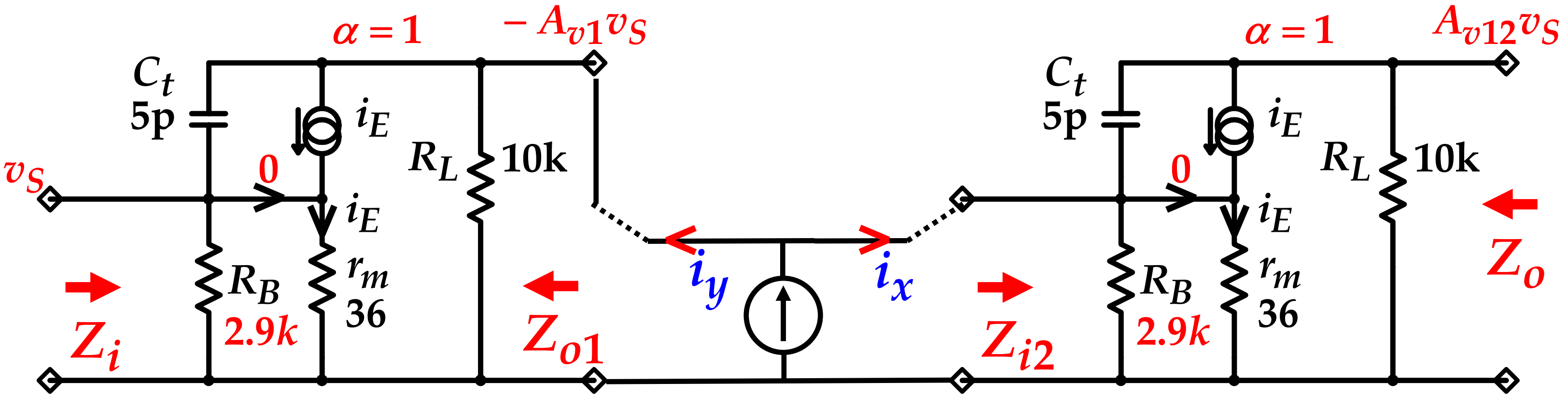
and $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$ = **current buffered gain of the 2 stages**

This result is actually the Chain Theorem (CT), and A_{v1}^{oc}, A_{v2}, Y_{t1}^{sc}, Z_{t2} are the (reciprocals of the) chain parameters (c parameters).

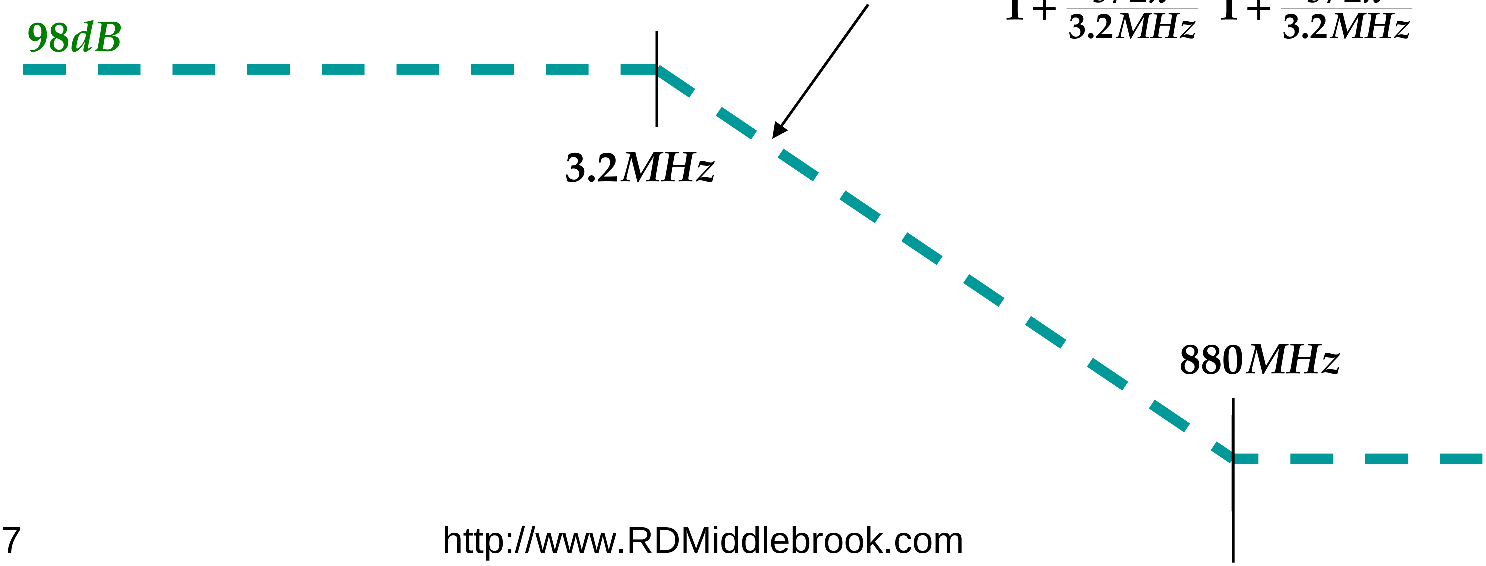
Rework the previous example:



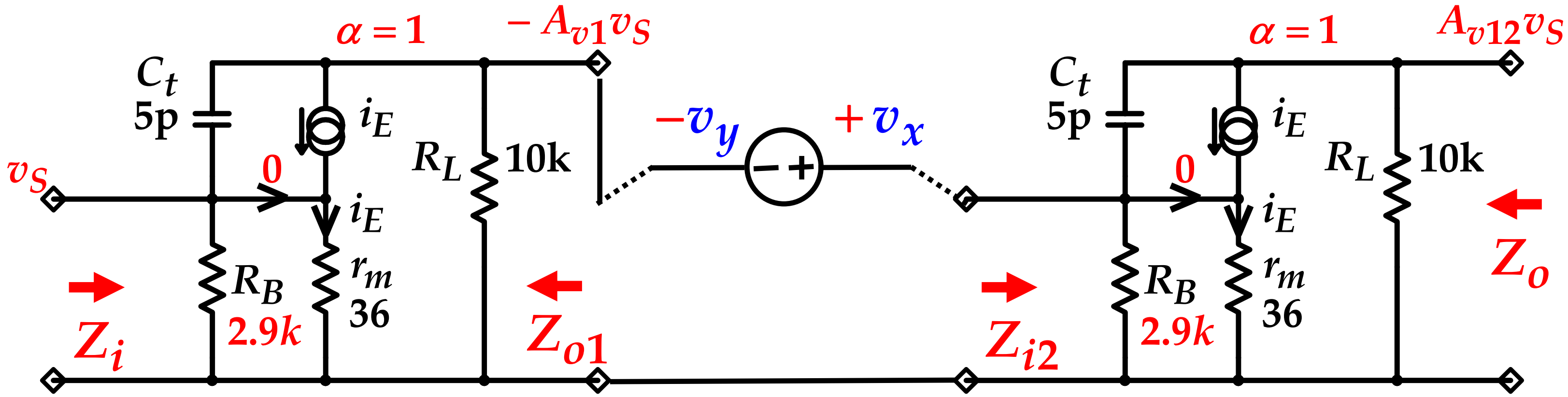
Rework the previous example:



$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

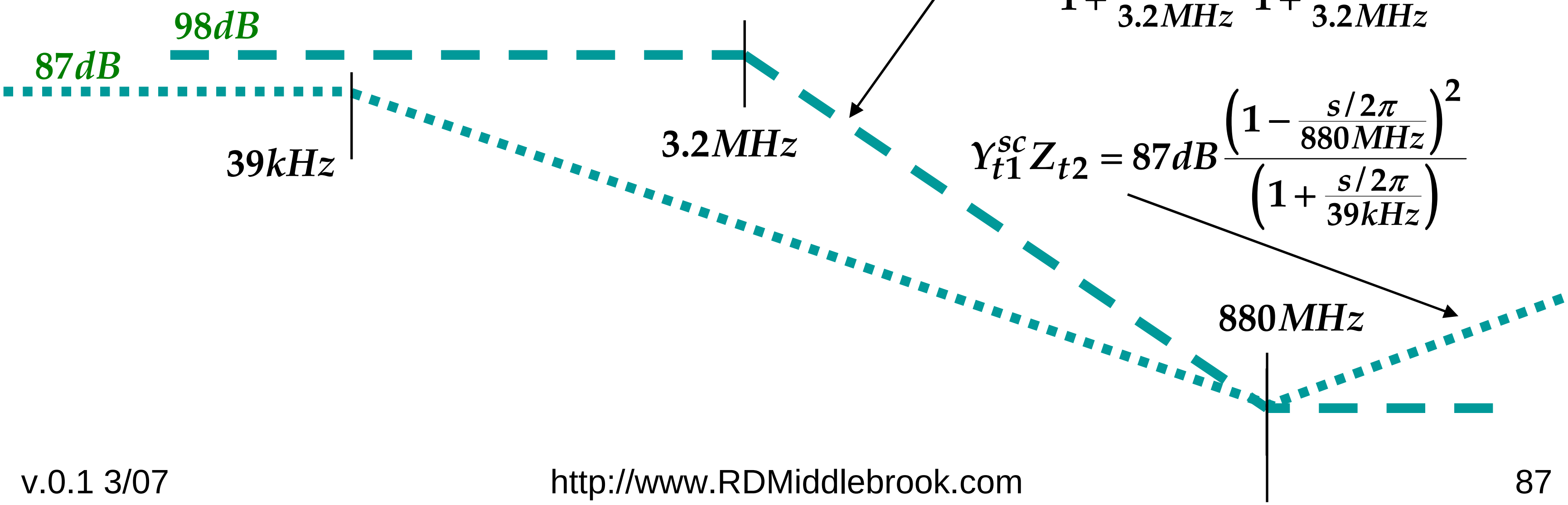


Rework the previous example:

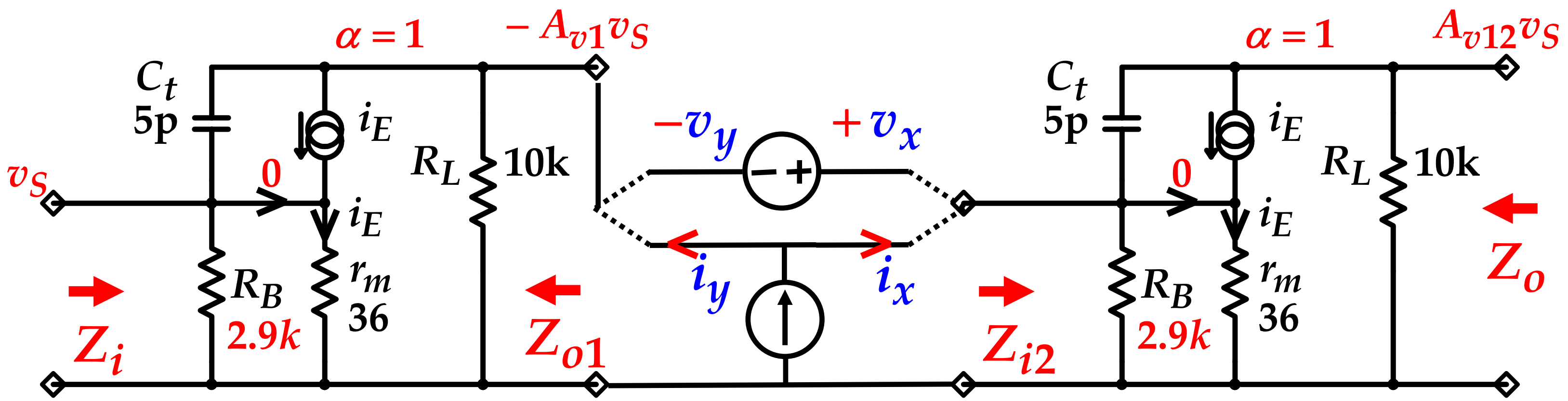


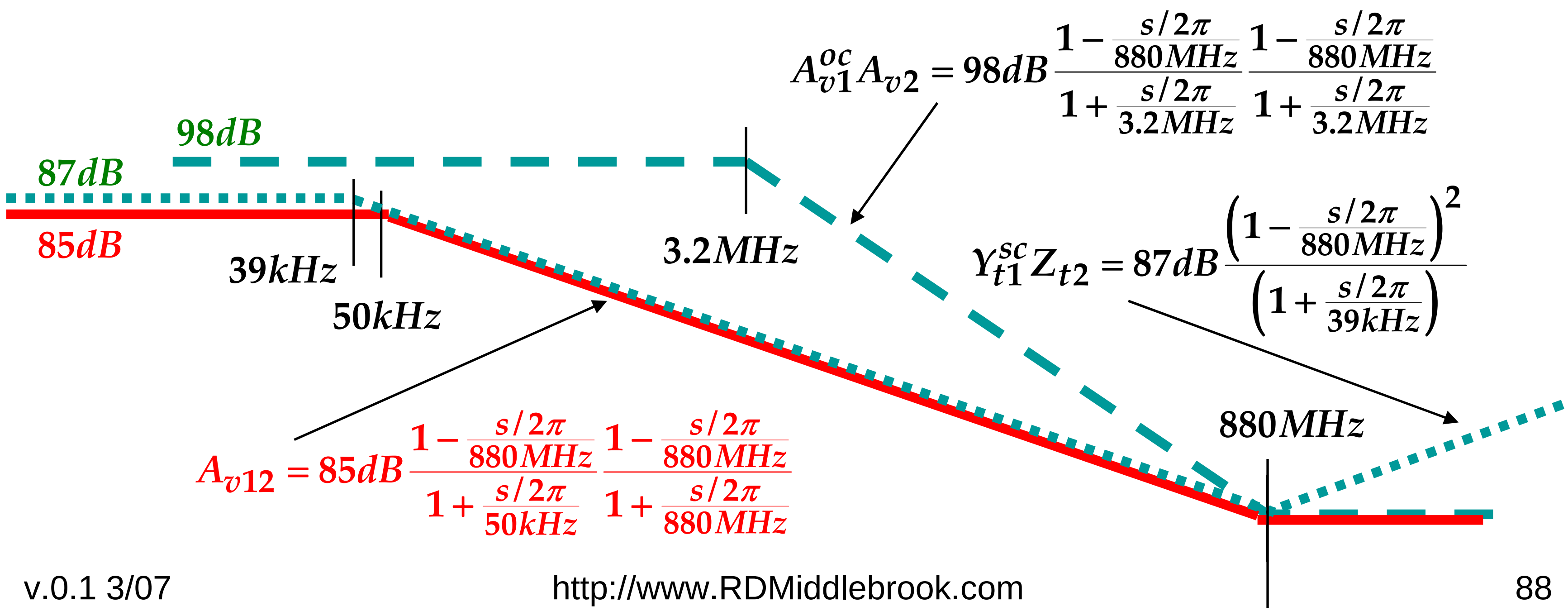
$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Y_{t1}^{sc}Z_{t2} = 87dB\,\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$



Rework the previous example:





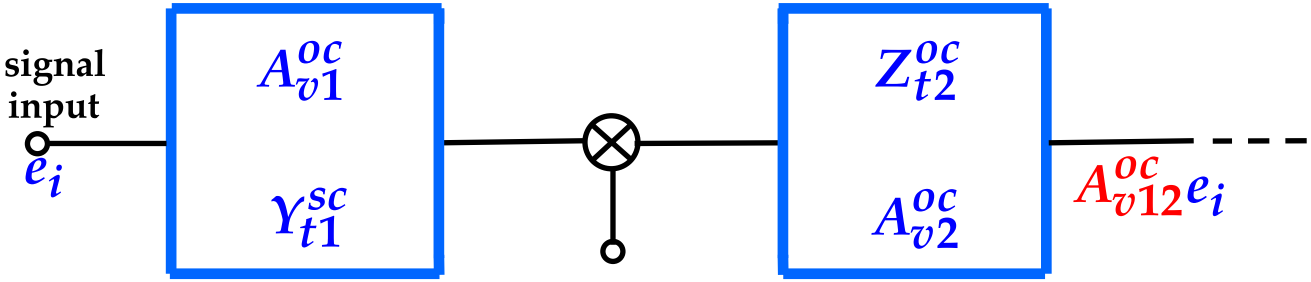
$$A_{v1}^{oc}A_{v2} = 98dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Y_{t1}^{sc}Z_{t2} = 87dB\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$

$$A_{v12} = 85dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}}\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

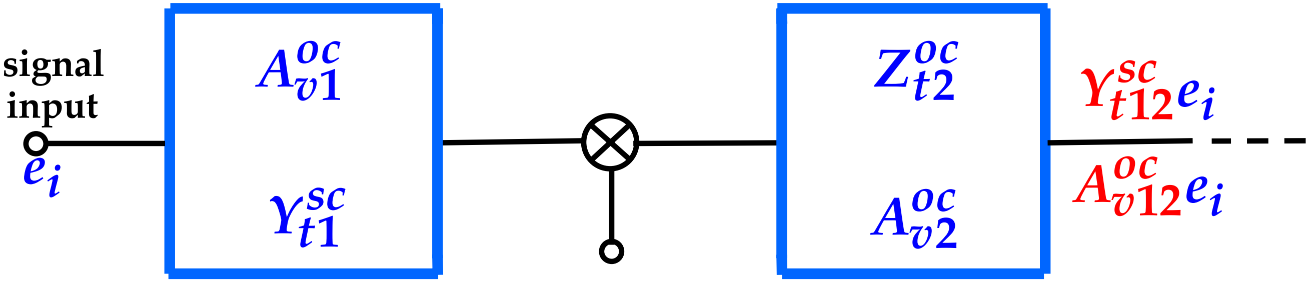
The procedure is:



Find A_{v1}^{oc} and Y_{t1}^{sc} of stage 1, and Z_{t2}^{oc} and A_{v2}^{oc} of stage 2. Combine them by the CT to find A_{v12}^{oc}, as above,

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

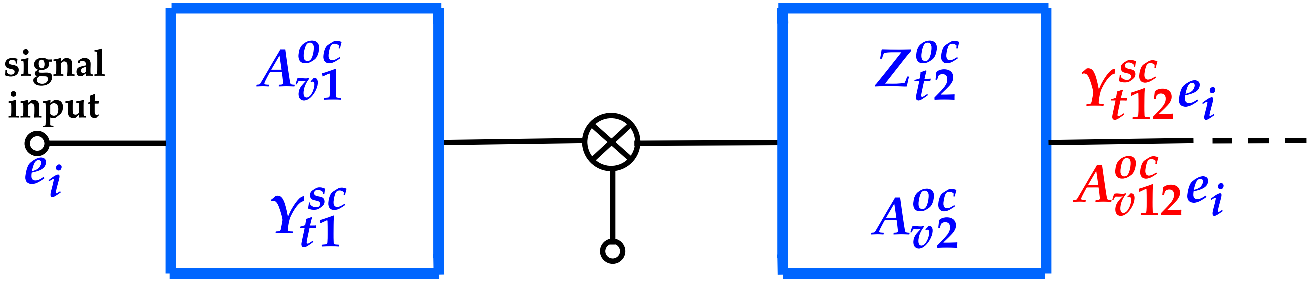
The procedure is:



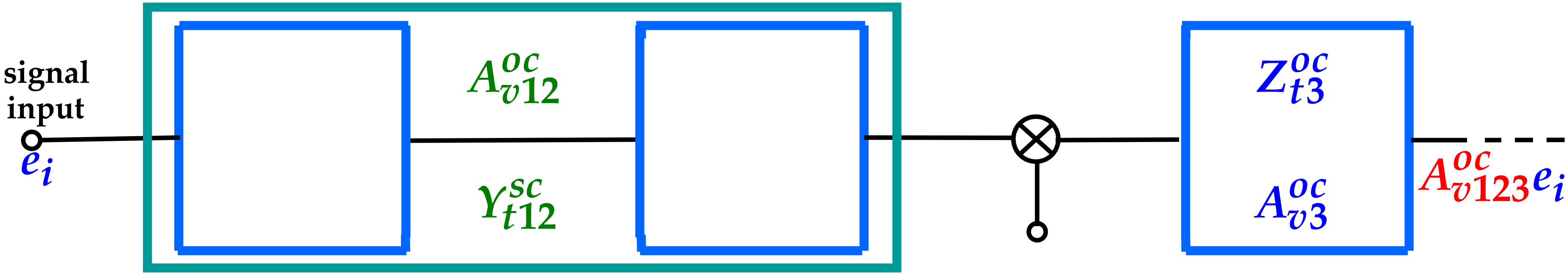
Find A_{v1}^{oc} and Y_{t1}^{sc} of stage 1, and Z_{t2}^{oc} and A_{v2}^{oc} of stage 2.
Combine them by the CT to find A_{v12}^{oc}, as above, and hence find Y_{t12}^{sc}.

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

The procedure is:



Find A^{oc}_{v1} and Y^{sc}_{t1} of stage 1, and Z^{oc}_{t2} and A^{oc}_{v2} of stage 2.
Combine them by the CT to find A^{oc}_{v12}, as above, and hence find Y^{sc}_{t12}.



Find Z^{oc}_{t3} and A^{oc}_{v3} of stage 3.
Combine them by the CT to find A^{oc}_{v123}.
and so on...